CYCLIC SQUARE ROOT OF GRAPHS

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Abstract

A graph G is said to have a cyclic square root if there exists a graph H which is cyclic and is such that $H^2 = G$ (upto isomorphism). A set of necessary and sufficient conditions for the existence of cyclic square roots for graphs is given. These conditions are in terms of cliques. An applications to 4-regular graphs is given.

Introduction

By a graph we mean a connected graph without loops and multiple edges.We follow the notations and terminology of Harary (1). The square of a graph G, written as G^2, is obtained by adding to G edges which join pairs of vertices of G at a distance 2 apart. A graph H is to be a square root of a graph G if $H^2 = G$ (up to isomorphism). Mukhopadhyay [3] presented a solution to the problem of characterising graphs having at least one square root graph. In this paper we obtain a set of necessary and sufficient conditions for the existence of a square root which is a cycle. We call such square roots as cyclic square roots..;

If G is a graph with at least three vertices, then a clique of G. A 3-clique is a clique on three vertices. If X is a finite set, then |X| will denote the number of elements of X. If a is a vertex of a graph G then the neighbourhood Va of a is $\{a\} \cup \{b \in V(G) \mid ab \in E(G)\}$. If S is a non -empty subset of V(G) then the induced subgraph of G by S, written G < S >, is the maximal subgraph of G with vertex set S. If we square a 3-cycle we get a 3-cycle and squaring the 4-cycles and 5- cycles yield the complete graphs on 4 and 5 vertices respectively. We therefore consider graphs with atleast 6 vertices.

Proposition : Let G be a non - complete graph which is the square of some cycle H, where $|V(H)| = |V(G)| \geq 6$. for

$a \in V(H) = V(G)$, let V_a denote the neighbourhood of a in H and $K_a = G < V_a >$; Then the following hold.

(i) K_a is a 3- clique in G, for each $a \in V(H)$

(ii) $V(K_a) \cap V(K_b) = \{a,b\}$ if and only if a is adjacent to b in H.

(iii)a is adjacent to c and c is adjacent to b in H if and only if $V(K_a) \cap V(K_b) = \{c\}$

(iv) $V(K_a) \cap V(K_b) = 0$ if and only if every path from a to b in H is of length ≥ 3.

(v) For each 3- clique in G there exist exactly two other 3- cliques each intersecting with it in exactly two vertices of G

Proof :

(i)Let $u, v \in V(K_a) = V_a$. The distance $d(u,v) \leq 2$ in H. Therefore $uv \in E(H^2) = E(G)$ and $u,v \in V(K_a)$. Hence k_a is complete. Now $a \in V(H)$ and H is a cycle. Therefore $|V(K_a)| = 3$. That K_a is maximal, follows from the fact that $|v(H)| \geq 6$.

Proofs of (ii),(iii)and (iv) are routine. We prove(v).

For $u_i \in V(H)$, by (i) $K_i = G < V_i >$ is a 3- clique where V_i is the neighbourhood of U_i in H. But H is a cycle. Therefore there exist u_j and U_k in V(H) such that u_i is adjacent to uj and U_k in H . Thus by (ii) $V(K_i) \cap V(K_j)$ contains exactly two vertices and $V(K_i) \cap V(K_k)$ also contains exactly two vertices. Therefore there exist two cliques K_j and K_k which intersect with K_i in exactly two vertices. Now suppose there exist a third clique different from K_j and K_k, say K_r such that $V(K_i) \cap V(K_r)$ contains exactly two vertices. Then u_i is also adjacent to u_r different from uj and u_k, that is, the degree of u_i is 3 or more in H, which is a contradiction, since H is a cycle. Therefore K_i meets two cliques in exactly two vertices.

Theorem 1: A non-complete graph G on p vertices $u_1, u_2, \ldots, u_p$ has a cyclic square root H if and only if there exists a collection of p 3-cliques K_1, K_2, ...K_p such that:

(a) $U_i \in V(K_i)$ for every i.

(b) $\bigcup_{i=1}^{P} E(K_i) = E(G)$.

(C) No two K_i s intersect in more than two vertices.

(d) For each K_i there exist exactly two other K_j and K_k such that K_i meets each of K_j and K_k in exactly two vertices.

(e) $u_i \in V(K_j)$ if and only if $u_j \in V(K_i)$ for every i and j.

Proof : Suppose G has a cyclic square root H. For each $u_i \in V(H)$, let V_i be the neighbourhood of u_i in H and $K_i = G <V_i>$. From proposition (i) each K_i is a 3-clique in G. Thus we have a collection of p 3-cliques in G. Now we have to show that at conditions (a) -(e) are satisfied. (a)is immediate from the definition of V_i (e) follows the fact that u_i is in the neighbourhood of u_j if and only if u_j is in the neighbourhood of ui in H. (c) and (d) follow from (ii),(iii), (iv) and (v) of proposition.

To prove (b) : let u_i uj $\in$ E $(G)=E(H^2)$. Therefore $d(u_i,u_j) \leq 2$ in H. Hence there exists $u_k \in V(H)$ such that u_i u_k u_j is a path in H. That is u_i $u_j \in E(K_k)$. Hence

$$E(G) \leq \bigcup_{i=1}^{P} (K_i).$$ The reverse inclusion is obvious.

Conversely, we define the graph H as.

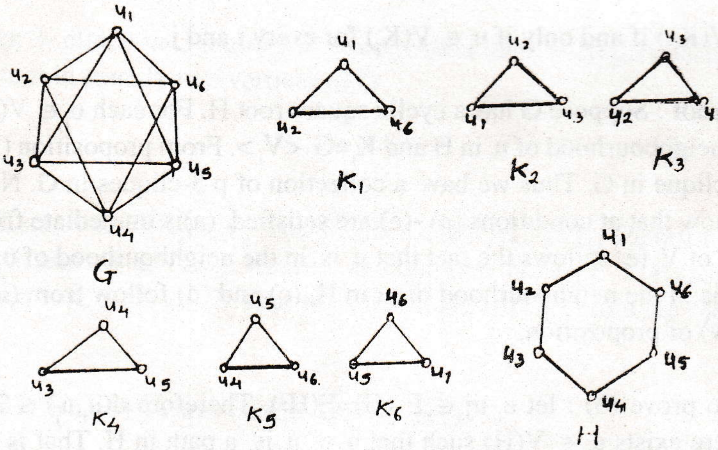
$V(H) = V(G) = \{u_1, u_2, ...u_p\}$ and
$E(H)$ $\{u_iu_j \in E(G) / V(K_i) \cap V(K_j) = \{u_i,u_j\}\}$ We show that
$E(H^2) = E(G)$. Let u_i u_j $\in E(H^2)$. If $d(u_i, u_j) = 1$ in H then $u_iu_j \in E(H) \leq E(G)$. Otherwise, there exists $u_k \in V(H)$ such that u_iu_j u_k u_j is a path in H. Therefore u_iu_j $V(K_k)$ and since Kk is complete, $u_iu_j \in E(K_k)$ for some k. By (b) u_i, $u_j \in$

$E(G)$. Thus $E(H^2) \leq E(G)$. Again let $u_i u_j \in E(G)$. From (b), (c) and (e)$V(K_i) \cap V(K_k) = \{u_i, u_k\}$ and $V(K_j) \cap V(K_k) = \{u_j, u_k\}$. Hence $u_i u_k \in E(H)$ and $u_j u_k \in E(H)$. That is $u_i u_k u_j$ is a path in H. Therefore $u_i u_j \in E(H^2)$, which proves that

$E(G) \leq E(H^2)$. Thus $E(G) = E(H^2)$. Also $|V(H)| = p$ and clearly H is connected. By (d) degree of each vertex of H is two. Hence H is a cycle and $H^2 = G$.

Remark. A procedure for finding all the cliques in a graph is known [2]. As a result, our Theorem gives a procedure for finding a cyclic square root of a 4- regular graph in which every clique is a 3- clique.

We illustrate the Theorem by an example .



Figure

A graph G and all its 3- cliques k_1 -k_6 are shown in figure. It is easily seen that the conditions of Theorem are satisfied by k_1 to k_6. The cycle H is a cyclic square root of G

Application:

Following theorem is an application of our Theorem 1.

Theorem 2: Any two 4- regular graphs with the same number of vertices having cyclic square roots are isomorphic.

Proof : Let G_1 be a 4- regular graph on p vertices $u_1, u_2, \ldots, u_p$ having cyclic square root c_1 say G_z be a 4- regular graph on p vertices $v_1, v_z, \ldots, v_p$ having cyclic square root c_z say. We can assume without loss that mapping f : $V(C_1) \rightarrow V(C_2)$ defined by $f(u_i) = v_i$ for each i is an isomorphism of c_1 onto c_z. We show that f preserves the adjacency and non-adjacency as a mapping of G_1 onto G_z. Let therefore u_i and u_j in G_1 . If u_i and u_j are adjacent in c_1 thenf(u_i) and f(u_j) are obviously adjacent in G_z. Suppose $d(u_i, u_j) = 2$ in C_1. A path of length 2 in c_1 is carried to a path of length 2in C_z by f. Hence f(u_i) and f(u_j) are adjacent in G_z. By symmetry, f preserves non-adjacency as well.

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