CATEGORY ANALOGUE OF TWO THEOREMS OF STEINHAUS

S. Basu *

Introduction

The distance between two vectors x and y in an n dimensional Euclidean space E^n is the non -negative real number || x-y || (with usual norm). Many papers have so far been devoted to the studies of those properties of sets E^n that are related to the distance between its vectors both for sets that are Lebesgue measurable ([2],[3]) and sets that have the property of Baire [1] . In this paper attempt has been made to prove the category analouges of the last two theorems of Steinhaus [3] (Theorem X and XI) in n -dimensional Euclidean space.

In theorem X, Steinhaus [3] proved that if $\{A_n\}_{n=1}^{\infty}$ is an infinite sequence of linear sets of positive measure, then here exists an infinite sequence of distinct points $\{a_n\}_{n=1}^{\infty}$ belonging respectively to A_n and such that their mutual distances are all rational .

Theorem XI of Steinhaus runs as follows :

If E is an infinite measurable subset (of the real line), then there exists an enumerable set P composed of points whose distances are rational and a set Z of measure zero such that $P \subseteq E \subseteq P' \cup Z$ where P' represents the derived set of P.

Theorem XI has been proved by using theorem X and some elementary properties of measurable sets.

Throughout this context, whenever the word 'set' has been used , it would mean a subset of E^n. For any set $A \subseteq E^n$ and for any vector $x_o \in E^n$, let us set $A_{(Xo)} = \{x + x_o \,/\, x \in A\}$. For any interval I we shall mean an n-

$I_{3(q2)}$ contains an open interval O_2 (say).

We next consider the set $O_2 \setminus F_{3(qo)}^{-(1)} \cup F_{2(q1)}^{-(2)} \cup F_{1(q2)}^{-(3)}$. On similar lines as above it can be shown that this set being non-empty open (in E^n) contains a non-degenerate compact interval J_3 (say). We set $Q_3^n = Q_2^n \setminus \{q_2\}$. Then the set Q_3^n being dense (in E^n), there exists $q_3 \in Q_3^n$ such that $J_3 \cap I_{4(q3)} \neq \phi$.

Likewise we begin our r-th stage with the set $O_{r-1} \setminus F_{r(qo)}^{-(1)} \cup F_{r-1(q1)}^{-(2)} \cup .... \cup F_{1(qr-1)}^{-(r)}$ where $q_{r-1} \in Q^n \setminus \{q_o ...., q_{r-2}\}$ $r \geq 2$ and O_{r-1} is an open interval (in E^n) contained in $J_{r-1} \cap I_{r(qr-1)}$.

If the above procedure is continued we obtain a sequence $J_1, J_2, ... J_K, ...$ of compact intervals (in E^n). Now from the fact that the intersection sequence of any decreasing sequence of non-empty compact sets is again non-empty, it follows that

$$\bigcap_{k=1}^{\infty} Jk \neq \phi.$$

But

$$\bigcap_{k=1}^{\infty} J_k \subseteq \bigcap_{j=1}^{\infty} (G_{1(q_o)} \setminus F_{j(q_o)}^{-(1)}) \cap \{ \bigcap_{k=2}^{\infty} \bigcap_{j=1}^{\infty} (G_{k(q_{k-1})} F_{k(q_{k-1})}^{-(k)}) \}$$

$$= (G_{1(q_o)} \setminus \bigcup_{j=1}^{\infty} F_{j(q_o)}^{-(1)}) \cap \{ \bigcap_{k=2}^{\infty} (G_{k(q_{k=1})} \setminus \bigcup_{j=1}^{\infty} F_{k(q_{k-1})}^{-(k)}) \}$$

$$\subseteq (G_{1(q_o)} \setminus \bigcup_{j=1}^{\infty} F_{j(q_o)}^{(1)}) \cap \{ \bigcap_{k=2}^{\infty} (G_{k(q_{k=1})} \setminus \bigcup_{j=1}^{\infty} F_{k(q_{k-1})}^{(k)}) \}$$

$$= (G_{1(q_o)} \setminus P_{1(q_o)}) \cap \{ \bigcap_{k=2}^{\infty} (G_{k(q_{k-1})} \setminus P_{k(q_{k-1})}) \}$$

$$\subseteq \bigcap_{k-1}^{\infty} \{(G_{k(q_{k=1})} \setminus P_{k(q_{k-1})}) \cup (P_{k(q_{k-1})} \setminus G_{k(q_{k-1})})\}$$

$$= A_{1(q_o)} \cap A_{2(q_1)} \cap ..... \cap A_{k(q_{k-1})} \cap .....$$

Hence the set $\bigcap_{k-1}^{\infty} A_{k(q_{k=1})} \neq \phi$

Therefore there exists a sequence $\{a_k\}_{k-1}^{\infty}$ of distinct vectors (in E^n) such that $a_k \in A_k$ (K=1,2,.....) and their mutual distances are rational numbers.

Using theorem 1 we now prove theorem 2.

Theorem -2: If A is a non-empty Baire set, then there exists an enumerable set P and a set H of first category such that the distance between any two vectors of P is an element of Q^n and $P \subseteq A \subseteq P' \cup H$ where P' is the derived set P.

Proof : If A is a set of first category, then choice of any enumerable set P (which may be finite) contained in A the mutual distances of whose elements are members of Q^n, will fulfill our purpose. So let us suppose that A is a Baire-set set of second category. We consider the sequence $I_1, I_2, \ldots, I_k, \ldots$ of open intervals whose *corners* are vectors representing elements of Q^n. From the sets $I_K \cap A$ (k=1,2....) we suppress those terms that represents sets of first category. We are therefore left with some sub sequence $I_{k_1} \cap A$, $I_{k_2} \cap A$,.......whose terms represents Baire sets of second category. Now using theorem-1, we find an infinite sequence $a_1, a_2, \ldots a_j$, ($a_j \in I_{k_j} \cap A$) of distinct vectors the mutual distance of any two of which is an element of thee set Q^n. We set $P = \{a_1, a_2, \ldots a_j, \ldots\}$ and H be the union of those sets that were suppressed. Clearly H is a set of first category and P is enumerable. Let $\xi \in A/H$. Since none of the sets form $I_k \cap A$ containing ξ has been suppressed there exists a sequence $I_{k_{j_1}}, I_{k_{j_2}}, \ldots I_{k_{j_r}}, \ldots$ of intervals such that

$\xi \in I_{k_{j_r}} \cap A$ (r=1,2,.....).

Hence it follows that $\xi = \lim_{r \to \infty} a_{k_{j_r}}$. Therefore $\xi \in p^f$ and hence $A \subseteq P' \cup H$. But $P \subseteq A$ Therefore $P \subseteq A \subseteq P' \cup H$ This completes the proof of he theorem.

Remark : If in the above theorem, A is given to be a Baire set of second category and we set

$$E = \bigcup_{j=1}^{\infty} (I_{k_j} \cap A)$$

Then we notice that there exists an enumerable set P the mutual distances of whose elements are rational numbers and a set H of first category such that A could be decomposed as $A = E \cup H$ where E is a Baire set of second category such that $P \subseteq E \subseteq P'$ 'where P' is the derived set of P.

At the end *we* like to mention that we may replace the set by any enumerable dense set which forms an additive sub -group of E^n with respect to the usual addition as the group operation.

In the conclusion, I wish to express deep gratitude to my guide, Dr. D.K. Ganguly for his helpful suggestions in preparation of this paper.

REFERENCES

1. J.C.Oxtoby, *Measure and Category*, Springer -Verlag, 1980.
2. S.Piccard, *Sur les ensembles de distance*, Memories Neuchatel University, 1938-39.
3. H. Steinhaus, *Surles distances des points dès ensembles de mesure positive*, Fund Math. 1(1920), 93-104.

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*C/O D.K. Ganguly, Dept. of Pune Mathematics , University of Calcutta .
35, Ballygunge Circular Road, Calcutta-700 019.