SOME PROPERTIES OF MATRIX TRANSFORMATION OF REAL SEQUENCES

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Introduction

Suppose that $\{u_n\}$ is a real sequence and $T=(a_{mn})$ be an infinite matrix with real constants. Let us consider

$$t_m = \sum_{n=1}^{\infty} a_{mn} u_n$$

where it is assumed that the right hand series is convergent for all m-1,2,........ .Then $\{t_m\}$ defines the T-transform of the sequence $\{u_n\}$. The sequence $\{u_n\}$ is said to be T-summable to s if t_n t ends to s as n tends to . A matrix $T=(a_{mn})$ is regular [1] if and only if

i) $\sup \sum_{n=1}^{\infty} |a_{mn}| < \infty$

ii) $\lim_{m\to\infty} a_{mn} = 0$ for every n

iii) $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = 1.$

If $a_{mn}=0$ for every $n> m$, the matrix is called triangular matrix. The matrix T reduces to Cesaro matrix if and $a_{mn} = 1/m$, $n \leq m$ and $a_{mn} = 0$, $n> m$.

Consider the sequences $\{u_n\}$ and define differences of different orders as follows

$$\Delta a_n = a_n - a_{m+1}$$

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_n - 2a_{n+1} + a_{n+2}$$

....

....

$$\Delta^r{}_{an} = \Delta(\Delta^{r-1} a_n) = a_n - {}^rc_1\, a_{n+1} + {}^rc_2\, a_{n+2} - \dots + (-1)^r a_{n+r},\ r \geq 2.$$

A sequence $\{u_n\}$ is said to be convex of order k-2 if $\Delta^k u_n > 0$ for all n and k $\geq$ 3 and it is simply convex if $\Delta^2 u_n > 0$ for all n.

A sequence $\{u_n\}$ is said to be of bounded variation if the series

$$\sum_{k=1}^{\infty} |u_k - u_{k+1}| \text{ is convergent}$$

In the present paper we study whether regular matrix transformation preserves convexity and bounded variation property of a sequence.

Theorem: 1 Cesaro matrix carries convex sequence and also convex sequence of order r-2(r $\geq$ 3) to a sequence of same nature.

Proof : Let $\{x_n\}$ be a convex sequence. Then Δ^2 xn > 0 for all n = 1,2,.......The transformed sequence is given by

$$z_n = \frac{1}{n} \sum_{k=1}^{n} x_k \quad \text{for all } n=1,2...$$

Now, $\Delta^2 z_n = z_n - 2z_{n+1} + z_{n+2}$

$$= \frac{1}{n} \sum_{k=1}^{n} x_k - \frac{2}{n+1} \sum_{k=1}^{n+1} x_k + \frac{1}{n+2} \sum_{k=1}^{n+1} x_k$$

$$= \left(\frac{1}{n} - \frac{n}{n+1} + \frac{2}{n+2}\right) \sum_{k=1}^{n} x_k - \left(\frac{2}{n+1} - \frac{1}{n+2}\right) x_{n+1} + \frac{1}{n+2} x_{n+2}$$

$$= \frac{1}{M}\{2 \sum_{k=1}^{n} x_k - n(n+3)\ x_{n+1}\ n(n+1)\ x_{n+2}\}$$

where $M = n(n+1)(n+2)$

$$= 1/M\{2(x_1 - 2x_2 + x_3) + 6(x_2 - 2x_3 + x_4) + \ldots + n(n+1(x_n - 2x_{n+1} + x_{n+2})\}.$$

$$= \frac{1}{M}\{2!\ \Delta^2 x_1 + 3!\ \Delta^2 x_2 + \ldots + \frac{(n+1)!}{(n-1)!} \Delta^2 x_n\}$$

Hence $\Delta^2 z_n > 0$ for all n=1,2......since $\Delta^2 x_n > 0$ for all n=1,2..... . So, convexity remains preserved under Cesaro matrix transformation . We shall now investigate he same for $\Delta^r x_n > 0$ for all n and for some $r(r \geq 3)$.

$$\Delta^r z_n = z_n - {}^rc_1\ z_{n+1} + {}^rc_2\ z_{n+2} \ldots + (-1)^r z_{n+r}$$

$$= \frac{1}{n} \sum_{k=1}^{n} x_k - {}^rc_1 \frac{1}{n+1} \sum_{k=1}^{n+1} x_k + {}^rc_2 \frac{1}{n+2} \sum_{k=1}^{n+2} x_k \ldots + (-1)^r . \frac{1}{n+r} \sum_{k=1}^{n+r} x_k$$

$$= [\frac{1}{n} - \frac{{}^rc_1}{n+1} + \ldots + \frac{(-1)^r}{n+r}] \sum_{k=1}^{n} x_k + [-\frac{{}^rc_1}{n+1} + \frac{{}^rc_2}{n+2} \ldots + \frac{(-1)^r}{n+r}] x_{n+1} + \ldots + \frac{(-1)^r}{n+r} x_{n+r}$$

$$= \frac{r!}{n(n+1) \ldots\ldots (n+r)} \sum_{k=1}^{n} x_k - \rho_1 x_{n+1} + \rho_2 x_{n+2} \ldots + (-1)^r \rho_r x_{n+r}$$

where $\rho_m = \sum_{i=m}^{r} \frac{{}^rc_i (-1)^{i+m}}{n+1}$, $1 \leq m \leq r$

$$= 1/M\{r!\ \Delta^r x_1 + (r+1)!\ \Delta^r x_2 + \ldots + \frac{(n+r-1)!}{(n-1)!} \Delta^r x_n\}$$

Therefore, $\Delta^r z_n > 0$ as $\Delta^r x_n > 0$ for all n.

Thus the theorem follows.

Note: We construct a regular matrix which can not transform a convex sequence to a convex sequence $\{x_n\}$ where $x_1 = 1$, $x_{n+2} = \{n(n+1)\}/2$ n = 0, 1,2.... .This is clearly a convex sequence. Consider the following regular matrix

$$A = \begin{vmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \ldots\ldots \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \ldots\ldots \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \ldots\ldots \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

It can be easily shown that A -transform of the sequence $\{x_n\}$ is not convex.

Theorem : 2 Infinite matrix transformation of Cesaro type of order 1 preserves bounded variation of a sequence.

To prove this theorem we need the following theorem [2] . A sequence $a=\{a_n\}$ is of bounded variation if and only if it can be written as a =b-c where $b=\{b_n\}$ and c $\{c_n\}$ are non -negative and non-increasing sequences.

Proof : Let $\{x_n\}$be a sequence of bounded variation. Then it can be expressed an $x_n = u_n - v_n$ where $\{u_n\}$ and $v\{V_n\}$ are non-negative and non -increasing sequences. Now the transformed sequence $\{z_n\}$ is given by

$$z_n = \frac{1}{n} \sum_{k=1}^{n} x_k \quad \text{for all } n = 1, 2 \ldots\ldots\ldots\ldots\ldots$$

Hence $$z_n = \frac{1}{n} \sum_{k=1}^{n} (u_k - v_k) = \frac{1}{n} \sum_{k=1}^{n} u_k - \frac{1}{n} \sum_{k=1}^{n} v_k = u'_k - v'_n \text{ (say)}$$

Since $u_n \geq u_{n+1}$' so $u_n - u_{n+1} \geq 0$ and also $u_i - u_{n+1} \geq 0$, $1 \leq i \leq n$ for all n.

Therefore,

$$u'_n - u'_{n+1} = \frac{1}{n} \sum_{k+1}^{n} u_k - \frac{1}{n+1} \sum_{k=1}^{n+1} u_k$$

$$= \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{k=1}^{n} u_k - \frac{1}{n+1} u'_{n+1} = \frac{1}{n(n+1)} \sum_{k=1}^{n} (u_k - u_{n+1}) \geq 0.$$ Thus $\{u'n\}$

is monotonic non-increasing sequence. Similar result holds for $\{u'_n\}$. so , z_n $u'_n - v'_n$ where $\{u'_n\}$and $\{v'_n\}$ are two monotonic non- increasing sequences . Hence $\{z_n\}$ is of bounded variation.

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