Mathematical Fourm
Vol XI, 1997

p.p. 16-20

COMPOSITION OPERATORS ON A LOCALLY CONVEX SPACE OF CONTINUOUS FUNCTIONS

BHOPINDER SINGH and **KAMALJEET KOUR**
(Department of Mathematics, University of Jammu, Jammu 180 004, India)

ABSTRACT

Let $C_{co}(X)$ be the locally convex space of all continuous complex (or real)–valued functions on a completely regular Hausdorff space X, which is endowed with its compact open topology. In this paper, we study compact composition operators on $C_{co}(X)$.

1. INTRODUCTION

Let X be a completely regular Hausdorff space and C(X) the vector space of all continuous complex (or real)–valued functions on X. For each compact subset K of X, we define the semi-norm p_K on C(X) as follows :

$p_K(f) = \sup \{|f(x)| : x \in K\}$.

Keywords and phrases : Locally convex space, compact-open topology, composition operators, compact operator.

The locally convex topology on C(X) generated by the seminorms $\{p_K : K \text{ compact in } X\}$ is called the compact-open topology of C(X). With this topology, C(X)[1] becomes a locally convex space and we denote it by $C_{co}(X)$.

Let F(X) be the vector space of all complex (real)–valued functions on X and let T be a selfmap on X. Then T induces the linear transformation $C_T : C_{co}(X) \rightarrow F(X)$, which is defined by

$C_T(f) = foT$ for all $f \varepsilon C_{co}(X)$.

We call C_T a composition operator on $C_{co}(X)$ if C_T is a continuous linear transformation from $C_{co}(X)$ into itself. These operators on different function spaces have been the subject matter of several papers in recent years, for example, see Kamowitz[2], Nordgren[3], Singh and Komal[4], Singh and Singh[6], and Takagi[8]. In this paper, first we study some properties of composition operators on $C_{co}(X)$, and then give the condition for a self map T on X to induce a compact composition operator C_T on $C_{co}(X)$.

2. RESULTS

We begin with the following characterization of composition operators on $C_{co}(X)$.

2.1 Proposition : A self map T on X induces a composition operator C_T on $C_{co}(X)$ if and only if T is continuous.

Proof : Suppose $T : X \rightarrow X$ is continuous. Then for every f in $C_{co}(X)$, $C_T f = foT \varepsilon C(X)$. Let K be an arbitrary compact subset of X. Then

$$\begin{aligned} p_K(C_T f) &= \sup \{|f(T(x))| : x \varepsilon K\} \\ &\leq \sup \{|f(y)| : y \varepsilon T(K)\} \\ &= p_{T(K)}(f) < \infty \end{aligned}$$

for every $f \varepsilon C_{co}(X)$. This shows that $C_T f \varepsilon C_{co}(X)$. To prove the continuity of C_T, let $\{f_\alpha\}$ be a net in $C_{co}(X)$ such that $f_\alpha \rightarrow O$ in $C_{co}(X)$. Then, for each compact subset K of X, $p_K(f_\alpha) \rightarrow O$. Now

$p_K(C_T f_\alpha) \leq p_{T(K)}(f_\alpha) \rightarrow O$, showing that C_T is continuous at origin and hence on $C_{co}(X)$. For the converse, suppose T is not continuous at some x_o in X. Then there is a net $\{x_\alpha\}_{\alpha\varepsilon\wedge}$ in X converging to x_o but $T(x_\alpha)$ does not converge to $T(x_o)$ in X. So there exists an open neighbourhood G of $T(x_o)$ such that for every $\alpha_o \, \varepsilon \wedge$, we have $T(x_\alpha) \not\in G$ for some $\alpha \geq \alpha_o$. Thus we can find a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $T(x_\beta) \not\in G$. Since X is completely regular, there exists an $f \, \varepsilon \, C_{co}(X)$ such that $O \leq f \leq 1$, $f(T(x_o)) = 1$ and $f(X/G) = \{O\}$. But since for each β,

$$|C_T f(x_\beta) - C_T f(x_o)| = |O - 1| = 1,$$

we have a contradiction to the fact that C_T is an operator on $C_{co}(X)$. Thus T must be continuous on X.

Examples : (i) Every constant function on X induces a composition operator on $C_{co}(X)$.

(ii) Every polynomial function on R induces a composition operator on $C_{co}(R)$.

(iii) If we define $T : R \to R$ on $T(x) = O$ when x is a rational, and $= 1$ if x is an irrational, then T is discontinuous at every point of R. Therefore, by Proposition 2.1, it does not induce a composition operator on $C_{co}(R)$.

Remark : (i) The set of all constant functions on X is invariant under C_T.

(ii) If X is a compact Hausdorff space, then $C_{co}(X)$ becomes a Banach space under the sup-norm : $\| f \|_\infty = \sup \{|f(x)| : x \, \varepsilon \, X\}$. In this case, if C_T is a composition operator on $C_{co}(X)$, then $\|C_T\| = 1$.

For x in X, if we define $e_x : C_{co}(X) \to C$ as $e_x(f) = f(x)$ for all f in Cco(X), then e_x is linear. Since $|e_x(f)| = |f(x)| \leq p_K(f)$ for some compact subset K of X, it follows that e_x is also continuous. Thus e_x is a continuous linear functional on $C_{co}(X)$. Let $E = \{e_x : x \, \varepsilon \, X\}$. Then E is a subset of $C_{co}(X)^*$, the continuous dual of $C_{co}(X)$. For each linear transformation H on $C_{co}(X)$, H induces the composition transformation C_H, which has the form

$C_H(A) = AoH$ for all $A \varepsilon C_{co}(X)^*$.

Next theorem presents a necessary and sufficient condition for an operator on $C_{co}(X)$ to be a composition operator. This result also holds in the general setting of weighted spaces of continuous functions (see, Singh and Summers[7]).

2.2 Theorem : Let $H : C_{co}(X) \to C_{co}(X)$ be an operator. Then $H = C_T$ for some self map T on X if and only if E is invariant under C_H.

Proof : It follows from Thoorem 3.1 of Singh and Summers[7].

Singh, Manhas and Singh[5] have proved a result for compact composition operators on weighted spaces of continuous functions, which in our setting of $C_{co}(X)$ yields the following :

2.3 Proposition : Let X be a connected completely regular Hausdorff space. Then a composition operator C_T on $C_{co}(X)$ is compact if and only if T is constant.

From their proof, we observe that they use connectedness of X only to establish that C_T is compact $\Rightarrow$ T is constant. Now, without using connectedness condition on X, we have the following characterization for a compact composition operator C_T on $C_{co}(X)$:

2.4 Theorem : Let X be a completely regular Hausdorff space and suppose T is a continuous self map on X. Then a composition operator C_T on $C_{co}(X)$ is compact if and only if T(X) is finite.

Proof : Let C_T be a compact composition operator on $C_{co}(X)$ and suppose there exists a sequence $\{x_n\}$ of distinct elements in T(X) which does not converge to any of its elements. Choose an open subset G_n of X such that $x_n \varepsilon G_n$ but $x_j \notin G_n$ when $j \neq n$.

Using complete regularity of X, there is a continuous function $f_n : X \to [0, 1]$ such that $f_n(x_n) = 1$ and $f(X/G_n) = \{O\}$ for all n. Now since

C_T is compact and $\{f_n\}$ is a bounded sequence in $C_{co}(X)$, $\{C_T f_n\}$ should have a convergent subsequence. But since for $j \neq n$ and for $x \in K$ with $T(x) = x_n$,

$$\begin{aligned} p_K(C_T f_n - C_T f_j) &= \sup\{|f_n(T(x)) - f_j(T(x))| : x \in K\} \\ &\geq |f_n(x_n) - f_j(x_n)| = 1, \end{aligned}$$

for each compact subset K of X, we obtain a contradiction. Thus T(X) must be finte. Conversely, if T(X) is finite, then it follows from Proposition 2.3 that C_T is compact.

REFERENCES

1. J. B. Conway, A Course in Functional Analysis, 2nd Ed. Springer-Verlag, New York, 1990.
2. H. Kamowitz, Compact Weighted Endomorphism of C(X), Proc. Amer. Math. Soc. 83(1981), 517–521.
3. E. A. Nordgren, Compsition Operators on Hilbert Spaces, Lecture Notes in Math. 693, Springer-Verlag, Berlin, 1978, p (37–63).
4. R. K. Singh and B. S. Komal, Composition Operators, Bull. Austral Math. Soc. 18(1978), 432–446.
5. R. K. Singh, J. S. Manhas and Bhopinder Singh, Compact Operators of Composition on Some Locally Convex Function Spaces, J. Operator Theory 31(1994), 11–20.
6. R. K. Singh and Bhopinder Singh, A Characterization of Weighted Composition Operators, Rocky Mountain J. Math. 23 (1993), 1107-1114.
7. R. K. Singh and W. H. Summers, Composition Operators on Weighted Spaces of Continuous Functions, J. Austral Math. Soc. (Series A) 45 (1988), 303–319.
8. H. Takagi, Compact Weighted Composition Operators on Certain Subspace of C(X, E), Tokyo J. Math. 14(1991), 121–127.