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LOOP ALGEBRAS OF MOUFANG LOOPS

W. B. VASANTHA KANDASAMY
Department of Mathematics
Indian Institute of Technology
Madras–600 036
India

In this note we initiate the study of loop algebras of Moufang loops. Loop algebras, which are loops over fields, are analogous to group algebras which are groups over fields ([2] and [3]). Here we study some properties of Moufang Loops.

[1] calls a non-empty set L to be a loop if L is endowed with the following properties :

(i) For all $a, b \varepsilon$L, $a \,.\, b \varepsilon$L where . is a binary operation from L x L $\rightarrow$ L.

(ii) For every ordered pair (a, b) ε L x L there is one and only one x such that $ax = b$ in L and only one y such that $ya = b$ in L.

(iii) There exists an element $e \varepsilon$L such that $ae = ea = a$ for every $a \varepsilon$L called the identity element of L.

Usually a loop is denoted by (L, . , e). We shall denote the identity element in L by 1. For further properties please refer [1].

Definition 1.

Let (L, . , 1) be a loop, K be a field. The loop algebra of the loop L over the field K with identity is the non-associative ring KL of all formal sums

$$\alpha = \sum_{m \varepsilon L} \alpha(m)m \text{ , } \alpha(m) \varepsilon K$$

such that supp $\alpha\{m / \alpha(m) \neq 0)$, the support of α is finite ; with the following operational rules.

1. $\sum_{m \varepsilon L} \alpha(m)m = \sum_{m \varepsilon L} \mu(m)m \Leftrightarrow \alpha(m) = \mu(m)$ for all $m \varepsilon L$

2. $\sum_{m \varepsilon L} \alpha(m)m + \sum_{m \varepsilon L} \mu(m)m = \sum_{m \varepsilon L} (\alpha(m) + \mu(m))m$

3. $\left(\sum_{m \varepsilon L} \alpha(m)m\right)\left(\sum_{m \varepsilon L} \mu(m)m\right) = \sum_{m \varepsilon L} \vartheta (m)m$

   where $\nu(m) = \sum_{xy = m} \alpha(x) \mu(y)$.

Dropping the zero components of the formal sum we may write

$\alpha = \sum_{i=1}^{n} \alpha_i m_i$. Thus $r \rightarrow r.1_m$ is an embedding of K in KL (where 1_m is the identity in L). After identification of K with $K.1_m$ we shall assume that K is contained in KL. Clearly $rm = mr$ for all $m \varepsilon L$ and $r \varepsilon K$. The element $1.1_m = 1$ acts as the identity of KL usually denoted by 1.

Remark.

Clearly KL is non-associative with respect to multipliction as L is non-associative under multiplication. Hence the major difference between a loop algebra and a group algebra is that loop algebra is non-associative where as a group algebra is associative.

[1] calls a loop L to be a Moufang loop if it satisfies any one of the following identities, then L has the inverse property and satisfies all the three conditions :

$$(xy)\,(zx) = [x(yz)]x\,,$$

$$[(xy)z]y = x[(y(zy)]\,,$$

$$x[y(xz)] = [(xy)x]z.$$

Further L satisfies the identities $(xx)y = x(xy)$, $(xy)x = x(yx)$, $(yx)x = y(xx)$.

Further every Moufang loop is diassociative and power associative [1].

Fur further properties please refer [1].

Theorem 2.

Let L be a finite Moufang loop and K any field. Then KL satisfies a polynomial identity.

Proof : Given L is a finite Moufang loop. Hence L is powerasociative and di-associative. Let $A = < x >$ be the subgroup generated by x as L is power associative and $G = < x, y >$ be the subgroup generated by x and y ($x \neq e$ $y \neq e$. e, identity of L) as L is di-associative. Now let $[G : A] = n < \infty$. Then by ([2] page 16) KL satisfies a standard polynomial identity of degree $2n$.

Proposition 3.

Let L be a Moufang Loop, K a field of characteristic O. The loop algebra KL contains nontrivial semi-prime rings.

Proof : Since L is Moufang, L is di-associative. Hence every pair of elements generates a subgroup. Let $G = < x, y >$. Then KG is semi-prime which is contained in KL (by [2]).

Problem.

What can one say about KL ? When will KL be semi-prime ?

Lemma 4.

Suppose L is a Moufang loop and K any field. KL is the loop algebra of L over F. Then there is a subgroup $G = < x, y > \subseteq L$ such that we have $K[\Delta] \subseteq KG \subseteq KL$.

Proof :

Since L is Moufang, L is di-associative, so let G = < x, y >, a subgroup of L. Since G is a group we can define

$$\Delta = \Delta G = \{x \varepsilon G \,/\, [G : C_G(x)] < \infty\} \subseteq L$$

Then $\Delta \subseteq G \subseteq L$, and so $K[\Delta] \subseteq K(G) \subseteq KL$.

Definition 5.

Let L be a Moufang Loop, K any field, and K[Δ] as in lemma 4. Let θ denote the projection

$$\theta : KL \to K[\Delta] \text{ given}$$

$$\text{by } \alpha = \sum_{m \varepsilon L} K_m m \to \theta(\alpha) = \sum_{x \varepsilon \Delta} K_x x$$

Then θ is a K linear map but not a ring-homomorphism.

Proposition 6.

Let L be a Moufang Loop, K a field of characteristic $p > 0$. Then the following are equivalent in every sub-group G generated by any two elements in L :

(i) KG is semi-prime

(ii) Δ (G) has no element of order p.

(iii) G has no finite normal subgroup with order divisible by p.

Proof Follows from [2].

Proposition 7.

Suppose L is a commutative Moufang loop which has no element of finite order, K any field then KL contains a non-trivial prime ring.

Proof :

L is diassociative.

Let $G = \langle x, y \rangle$ be the group generated by x and $y \varepsilon L$. Since G is torsion free abelian, $\Delta(G)$ is also torsion free abelian. Hence by ([2] page 6) KG is a non-trivial prime ring contained in KL.

Problem.

Can KL in proposition 7 have non-trivial divisors of zero ?.

Proposition 8.

Suppose L is a finite Moufang loop, K any field. Then KL contains non-trivial divisors of zero.

Proof :

Since L is di-associative, let $G = \langle x, y \rangle$ be the subrgoup generated by x and $y \varepsilon G \subseteq L$. G is finite as L is finite. So $KG \subseteq KL$ has nontrivial divisors of zero.

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