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HYDROMAGNETIC COUETTE FLOW IN A ROTATING SYSTEM IN THE PRESENCE OF AN INCLINED MAGNETIC FIELD.

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The hydromagnetic Couette flow in a rotating system in the presence of a uniform magnetic field inclined at an angle θ with the positive direction of the axis of rotation is considered. It is found that the primary velocity increases while the secondary velocity decreases with increase in the angle of inclination of the magnetic field. It is also found that for large rotations, there exists a thin boundary layer near the plates which increases with the increase in θ.

1. INTRODUCTION

The problem of magnetohydrodynamic Couette flow is of general interest. Several authors [1–3] have studied various aspect of the problem. The hydromagnetic Couette flow in a rotating frame of reference have been studied by Jana and Datta [4]. In all these studies, the applied magnetic field is taken in the transverse direction of the flow field. The hydromagnetic Couette flow in a rotating system in the presence of a magnetic field inclined with the axis of rotation has not received much attention. The study of such fluid flow problem is important because of its wide application in various branches of geophysics, astrophysics and fluid engineering.

In the present paper, we have studied the hydromagnetic Couette flow in a rotating system in the presence of a uniform magnetic field inclined at an angle θ with the positive direction of the axis of rotation. An exact solution of the governing equations for the fully—developed flow is obtained. It is found that for large values of the rotation parameter K^2, there exist a thin boundary layer near the plates. The thickness of the boundary layer increases with increase in angle of inclination θ of the magnetic field. It is also observed that for small values of K^2 and the Hartmann number M^2, the primary velocity u_1 is independent of K^2 while the secondary velocity v_1 is independent of M^2 for all values of θ.

2. MATHEMATICAL FORMULATION AND ITS SOLUTION

Consider the steady flow of a viscous incompressible electrically conducting fluid between two infinite parallel plates separated by a distance h. The lower plate moves with a uniform velocity U in the x-direction where the x-axis is taken on the lower plate. The z-axis perpendicular to the plates and y-axis normal to xz-plane. The fluid and the plates are to rotate in unison with constant angular velocity Ω about z-axis. A uniform magnetic flux B_O acts along z-axis which is inclined at an angle θ with the positive direction of z-axis. Since the plates are infinite long along x-and y-direction, all physical quntities will be function of z only. The following assumptions are compatible with the fundamental equations of magnetohydrodynamics

$$\vec{q} = (u, v, o), \quad \vec{B} = (B_x + B_O \operatorname{Sin}\theta, B_y, B_O \operatorname{Cos}\theta),$$
$$\vec{E} = (Ex, Ey, Ez), \quad \vec{j} = (j_x, j_y, o), \tag{1}$$

where $\vec{q}$, $\vec{B}$, $\vec{E}$, and $\vec{j}$ are respectively the velocity vector, the magnetic induction vector, the electric field vector and current density vector. In general, the electric current flowing in the fluid distorts the applied magnetic field. However, since the viscous boundary layer is thin and thermally ionized air is at best a poor electrical conductor, it is permissible to neglect the induced magnetic field compared to the applied one [see Pai [5]) so that $\vec{B} \equiv (B_O \operatorname{Sin}\theta, o, B_O \operatorname{Cos}\theta)$. Further it is assumed that no applied or polarization voltage exists i.e. $\vec{E} = O$.

Under the above assumptions, the hydromagnetic equations of motion in a rotating frame of reference, in dimensionless form become

$$-2\,K^2 v_1 = \frac{d^2 u_1}{d\eta^2} - M^2 u_1 \operatorname{Cos}^2\theta, \tag{2}$$

$$2\,K^2 u_1 = \frac{d^2 v_1}{d\eta^2} - M^2 v_1, \tag{3}$$

where

$$u_1 = u/U,\; v_1 = v/U,\; \eta = z/h,\; K^2 = \Omega h^2/\nu,\; M^2 = \sigma B_0^2\, h^2/\rho\nu, \tag{4}$$

The solution of the equations (2) and (3) subject to the boundary conditions (5) are

$$u_1 = \frac{1}{4i\alpha\beta}\left[\left\{(\alpha + i\beta)^2 - M^2\right\} \frac{\operatorname{Sinh}(\alpha + i\beta)(1-\eta)}{\operatorname{Sinh}(\alpha + i\beta)} - \left\{(\alpha - i\beta)^2 - M^2\right\} \frac{\operatorname{Sinh}(\alpha - i\beta)(1-\eta)}{\operatorname{Sinh}(\alpha - i\beta)}\right] \tag{6}$$

$$v_1 = \frac{2K^2}{4i\alpha\beta}\left[\frac{\operatorname{Sinh}(\alpha + i\beta)(1-\eta)}{\operatorname{Sinh}(\alpha + i\beta)} - \frac{\operatorname{Sinh}(\alpha - i\beta)(1-\eta)}{\operatorname{Sinh}(\alpha - i\beta)}\right] \tag{7}$$

where

$$\alpha, \beta = \frac{1}{2}\left[\left\{M^4\left(1 + \operatorname{Cos}^2\theta\right)^2 + 4k_1^4\right\}^{1/2} \pm M^2\left(1 + \operatorname{Cos}^2\theta\right)\right]^{1/2},$$

$$K_1^2 = \left(16K^4 - M^4 \operatorname{Sin}^4\theta\right)^{1/2}. \tag{8}$$

The above solution is valid for $\operatorname{Sin}\theta < 4K^2/M^2$.

We shall now discuss the following particular cases.

Case (i) : When $K^2 << 1$ and $M^2 << 1$.

Since K^2 and M^2 are very small, by neglecting higher order of K^2 and M^2 in (6) and (7), we get

$$u_1 = 1 - \eta - \frac{1}{6} M^2 \operatorname{Cos}^2\theta \, (2\eta - 3\eta^2 + \eta^3) + \ldots, \tag{9}$$

$$v_1 = -\frac{1}{3} K^2 (2\eta - 3\eta^2 + \eta^3) + \ldots \tag{10}$$

It is seen from above equations (9) and (10) that in a slowly rotating system when the applied magnetic field is weak, the primary velocity u_1 is independent of rotation while the secondary velocity v_1 is unaffected by the magnetic field for all values of θ.

Case (ii) : When $M^2 >> 1$ and $K^2 << 1$.

In this case neglecting higher powers of K^2 in the equations (6) and (7), we obtain

$$u_1 = \left[1 + \frac{M^2 \operatorname{Sin}^2\theta}{2\sqrt{2M^2 (1 + \operatorname{Cos}^2 \theta)}} \eta\right] \exp\left[-\left\{\frac{1}{2} M^2 (1 + \operatorname{Cos}^2 \theta)\right\}^{\frac{1}{2}}\right] \eta \tag{11}$$

$$v_1 = -\frac{2K^2}{\sqrt{2M^2 (1 + \operatorname{Cos}^2 \theta)}} \eta \exp\left[-\left\{\frac{1}{2} M^2 (1 + \operatorname{Cos}^2 \theta)\right\}^{\frac{1}{2}}\right] \eta \tag{12}$$

The above equations (11) and (12) show the existance of a single—deck boundary layer of thickness of order

$$o\left(\frac{\sqrt{2}}{M (1 + \operatorname{Cos}^2\theta)^{1/2}}\right)$$

near the plates. This boundary layer is known as modified Hartmann boundary layer. The thickness of this boundary layer increases with increase in the angle of inclination θ of the magnetic field while it decreases with increase in Hartmann number M. Further, the modified Hartmann boundary layer thickness is independent of rotation parameter K^2.

Case (iii) : When $K^2 >> 1$ and $M^2 << 1$.

Since M^2 is small, neglecting higher powers of M^2 in (6) and (7), we have

$$u_1 = \left(\frac{M^2 \sin^2\theta}{4K^2} \sin\beta\eta + \cos\beta\eta\right) \bar{e}\ \alpha\eta, \tag{13}$$

$$v_1 = -\bar{e}\alpha\eta \sin\beta\eta, \tag{14}$$

where

$$\alpha, \beta = K\left[1 \pm \frac{M^2 (1 + \cos^2\theta)}{8K^2}\right]. \tag{15}$$

It is seen from equations (13) and (14) that there exist a single—deck boundary layer near the plates, the thickness of the boundary layer is of order $O(1/\alpha)$ where α is given by (15). This boundary layer is known as Ekman—Hartmann boundary layer, The thickness of this layer increases with increase in either θ or K^2 while it decreases with increase in M^2.

Case (iv) : Single plate motion :

In the limit $h \to \infty$, on introducing $\eta = Uz/v$, $M^2 = \sigma B^2 U/\rho v$, $K^2 = \Omega v/U^2$, the velocity components u_1 and v_1 given by (6) and (7) are reduced to

$$u_1 = \frac{K^2}{4i\alpha_1\beta_1}\left[\left\{\left(\alpha_1 + i\beta_1\right)2 - M^2\right\} \bar{e}^{(\alpha_1 + i\beta_1)\eta}\right.$$

$$\left. - \left\{\left(\alpha_1 - i\beta_1\right)^2 - M^2\right\} \bar{e}^{(\alpha_1 - i\beta_1)\eta}\right\}\Big], \tag{16}$$

$$v_1 = \frac{K^2}{2i\alpha_1\beta_1}\left[\bar{e}^{(\alpha_1 + i\beta_1)\eta} - \bar{e}^{(\alpha_1 - i\beta_1)\eta}\right], \tag{17}$$

where

$$\alpha_1, \beta_1 = \frac{1}{2}\left[\left\{M^4 (1 + \cos^2\theta)^2 + 4K_1^4\right\}^{\frac{1}{2}} \pm M^2 (1 + \cos^2\theta)\right]^{\frac{1}{2}}, \tag{18}$$

$$K_1^2 = (16K^4 - M^4 \sin^4\theta)^{1/2}.$$

The equations (16) and (17) indicate that in the limit $h \to \infty$, the fluid velocities corresponds to a single plate problem when the plate moves in its own plane with a uniform velocity U in the presence of a uniform magnetic field inclined at an angle θ with the positive direction of the axis of rotation. Here, also, exists a thin boundary layer in the vecinity of the plate and the thickness of this layer is of order $O(1/\alpha_1)$ where α_1 is given (18).

If $\theta = O$, then the equations (16) and (17) reduce to

$$u_1 = \bar{e}^{\alpha_1 \eta} \operatorname{Cos} \beta\eta \text{ and } v_1 = \bar{e}^{\alpha_1 \eta} \operatorname{Sin}\beta_1 \eta \tag{19}$$

where

$$\alpha, \beta = \frac{1}{\sqrt{2}} \left[\left(M^4 + 16K^4 \right)^{1/2} \pm M^2 \right]. \tag{20}$$

These are the velocity components which correspond to the single plate motion when the plate moves in its own plane with a velocity U in the presence of a uniform magnetic field transverse to the flow field.

3. RESULTS AND DISCUSSIONS :

The distribution of the primary and the secondary velocity components are plotted against η for $M^2 = 4.0$ and for different values of θ and K^2 in figures 1 to 3. It is seen from fig. 1 that for fixed values of M^2 and K^2 the primary velocity increases while the secondary velocity decreases with increase in the angle of inclination θ of the magnetic field. It is observed from Figures 2 and 3 that both the primary and the secondary velocities decrease with increase in rotation parameter K^2 when θ is fixed. It is also observed that for large values of K^2, there is an incipient flow reversal near the stationary plate.

The non—dimensional shear stresses at the plate $\eta = O$ are given by

$$\tau_x = -\left.\frac{du_1}{d\eta}\right)_{\eta = O} = \frac{d_2 \operatorname{Sin}h\, 2\alpha + C_2 \operatorname{Sin} 2\beta}{2\alpha\beta\, (\operatorname{Cosh} 2\alpha - \operatorname{Cos} 2\beta)}, \tag{21}$$

$$\tau_y = -\left.\frac{dv_1}{d\eta}\right)_{\eta = 0} = \frac{K^2 (\beta \operatorname{Sinh} 2\alpha - \alpha \operatorname{Sin} 2\beta)}{\alpha\beta (\operatorname{Cosh} 2\alpha - \operatorname{Cos} 2\beta)}, \tag{22}$$

where

$$C_1 = \alpha^2 - \beta^2 - M^2,\ C_2 = \alpha c_1 - \beta d_1,$$

$$d_1 = 2\alpha\beta,\ d_2 = \beta c_1 + \alpha d_1. \tag{23}$$

The numerical values of τ_x and τ_y are shown in figure 4 against θ for different values of K^2. It is found that the shear stress due to the primary flow decreases while that due to the secondary flow increases with increase in θ when K^2 is fixed. On the other hand for fixed value of θ, both the shear stresses increase with increase in K^2.

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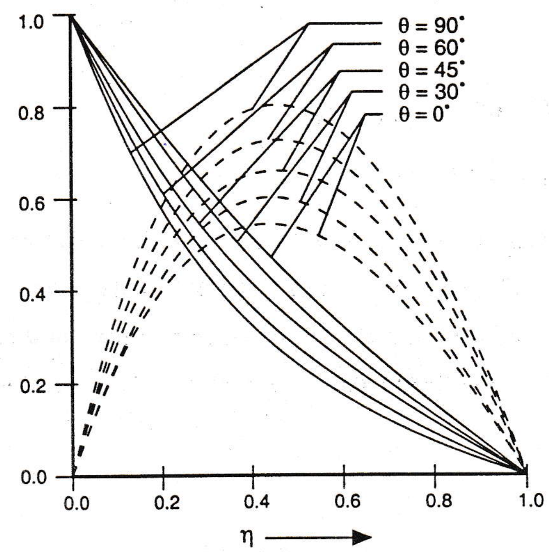

Fig. 1 Velocity components ——— u and - - - - -v for k^2 = 2.0 and M^2 = 4.0

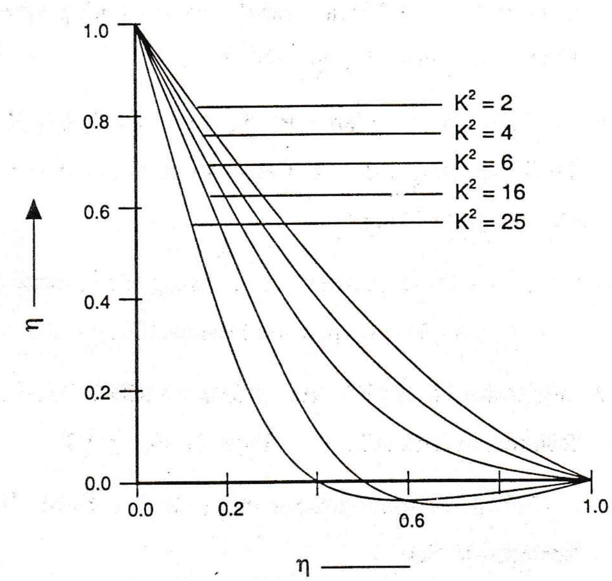

Fig. 2 Velocity u for M^2 = 4.0 and θ = 45°

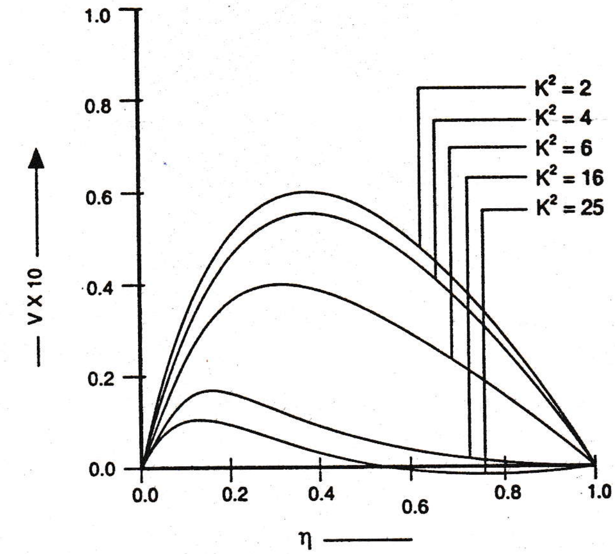

Fig. 3 Velocity — v x 10 for $M^2 = 4.0$ and $\theta = 45^\circ$

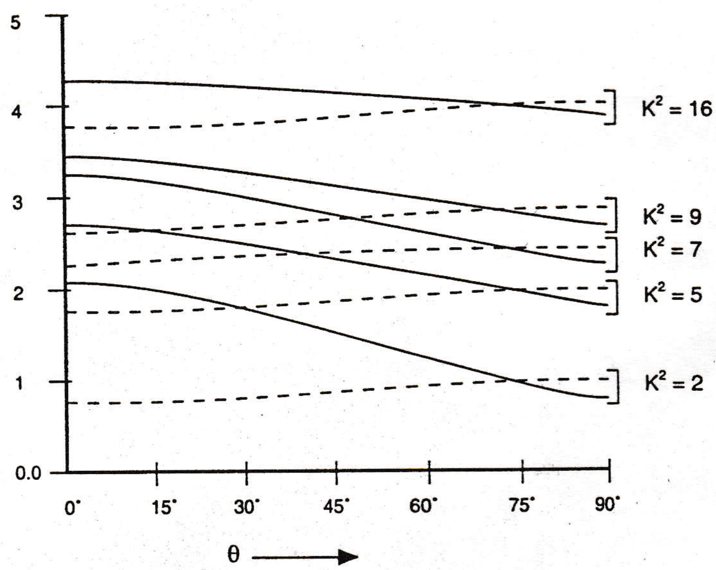

Fig. 4 Shear stress components ———— τ_x and ———— τ_y for $M^2 = 2.0$