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NONLOCAL CONTROLLABILITY OF NONLINEAR INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACE

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ABSTRACT

In this paper a new type of controllability is introduced for dynamical systems represented by integrodifferential equations in Banach space. Several sufficient conditions are established for such controllability to nonlinear integrodifferential evolution systems. The results are obtained using the semigroup of linear operators and the Schauder fixed point theorem.

Key Words : *Controllability, Integrodifferential equation, Fixed point theorem, Nonlinear delay systems.*

AMS (MOS) Subject Classification : 93 B 05.

1. INTRODUCTION

It is well known that the problem of controllability of dynamical systems represented by differential equations in finite dimensional space is nothing but the two point boundary value problem [10, 14]. This problem has been studied by several authors and extended to infinite dimensional spaces for various kinds of systems including nonlinear delay systems [6, 7] and integrodifferential systems [1]. The motivation to study the problem with nonlocal boundary conditions is that the theory of diffussion and heat conduction, the nonlocal boundary condition has better effects than the analogous known classical parabolic problems (see [1, 2]). Several authors [3, 4, 13] have given importance to this area of study and also discussed the problems using nonlocal boundary conditions. Based on these recent investigations we formulate a new type of controllability problem, by utilizing the nonlocal boundary conditions. Here the control function u steers the solution of the nonlocal boundary value problem from the initial state to the final state both having nonlinear function g. For application, the controllability of parabolic and hyperbolic nonlinear problems together with nonlocal conditions are studied.

The purpose of this paper is to study the new type of controllability for nonlinear integrodifferential systems with nonlocal boundary conditions by using the Schauder fixed point theorem. The considered nonlinear evolution equations serves as an abstract formulation of many parabolic and hyperbolic differential equations which arise in the problems with heat-flow, epidemic and other physical phenomenon [8, 9, 11, 13].

2. PRELIMINARIES

Consider the nonlinear integrodifferential system

$$\frac{dx}{dt}(t) + Ax(t) = Bu(t) + f(t,x(t)) + \int_0^t G\Big(t,s,x(s), \int_0^s K(s,\tau,x(\tau))d\tau\Big)ds, \quad t\in J = [0,T] \quad (1)$$

with nonlocal boundary condition of the form

$$x(0) + g(x(t_1), x(t_2), \ldots, x(t_p)) = x_o \quad (2)$$

where $0<t_1<t_2,\ldots,<t_p<T$, $p \in N$, the state $x(t)$ takes the values in the Banach space E and the control function u is given in $L^2(J,V)$, a Banach space of admissible control functions with V a Banach space. Here linear operator $-A$ is the infinitesimal generator of a C_o semigroup $U(t)$, $t \geq 0$ on a Banach space E, B is a bounded linear operator from V into E. The nonlinear operators

$K : JxJxE \to E$, $G : JxJxExE \to E$, $f : JxE \to E$ and $g : E^p \to E$ are uniformly bounded and continuous.

For the system (1) there exists mild solutions of the following forms [5, 12]

$$x(t) = U(t)x_o - U(t)g(x(t_1),x(t_2),\ldots,x(t_p))$$

$$+\int_0^t U(t-s)\left[(Bu)(s) + f(s,x(s))\right]ds + \int_0^t U(t-s)\left[\int_0^s G(s,\tau,x(\tau),Q(\tau))d\tau\right]ds \quad (3)$$

where $Q(t) = \int_0^t K(t,s,x(s))ds$.

The nonlocal condition (2) can be applied in physics with better effect than classical condition $x(0) = x_o$ since the condition (2) is usually more precise for physical measurements than condition $x(0) = x_o$ (see[5]). Based on the above observation we are introducing the following type of controllability in this work.

Definition : The system (1) is said to be **nonlocally controllable** on the interval J if, for every x_o, x_1 and g are all in E, there exists a control $u \in L^2(J,V)$ such that the solution $x(t)$ of (1) satisfies

$$x(0) + g(x(t_1), x(t_2),\ldots,x(t_p)) = x_o$$

and

$$x(T) + g(x(t_1), x(t_2),\ldots,x(t_p)) = x_1$$

Next we shall establish the controllability result for the nonlinear system (1). For that we assume the following hypotheses :

(i) the linear operator $-A$ generates a compact semigroup $U(t)$ such that

$\max_t \| U(t) \| \leq M_1$, where M_1 is a positive constant.

(ii) the linear operator W from V into E defined by

$$Wu = \int_0^T U(T-s)Bu(s)ds \qquad (4)$$

has an invertible operator W^{-1} defined $L_2(J,V)\backslash\ker W$ and there exist positive constants M_2 and M_3 such that $\|B\| \leq M_2$ and $\|W^{-1}\| \leq M_3$

(iii) the nonlinear operators $f : JxE \rightarrow E$, $G : JxJxExE \rightarrow E$ and $g : E^p \rightarrow E$ satisfy the conditions

$$\|g(x(t_1), x(t_2),\ldots,x(t_p))\| \leq K_1.$$

$$\|f(t,x(t))\| \leq K_2,$$

and $$\|G(t,s, x(s), Q(s))\| \leq K_3, \text{ where } K_1, K_2, K_3 > 0.$$

3. MAIN RESULTS

THEOREM 3.1 : If the hypotheses (i) - (iii) are satisfied, then the system (1) is nonlocally controllable on J.

PROOF : Using the hypothesis (ii), define the control

$$u(t) = W^{-1}\Big\{ x_1 - g(x(t_1), x(t_2),\ldots,x(t_p)) - U(T)x_o + U(T)g(x(t_1), x(t_2),\ldots,x(t_p))$$

$$- \int_0^T U(T-s)\Big[f(s, x(s)) + \int_0^s G(s,\tau, x(\tau), Q(\tau))d\tau\Big]ds\Big\}(t) \qquad (5)$$

Then from (3) we have

$$x(T) = x_1 - g(x(t_1), x(t_2),\ldots,x(t_p))$$

We shall now show that, when using this control, the operator Φ defined by

$$(\Phi x)(t)=U(t)x_o - U(t)g(x(t_1), x(t_2),\ldots,x(t_p)) + \int_0^t U(t-s)\Big[(Bu)(s) + f(s,x(s))\Big]ds$$

$$+\int_0^t U(t-s)\left[\int_0^s G(s,\tau,x(\tau),Q(\tau))\,d\tau\right]ds \tag{6}$$

has a fixed point. This fixed point is then a solution of (3).

Clearly $$(\Phi x)(0) + g(x(t_1), x(t_2),\ldots,x(t_p)) = x_o$$

and $$(\Phi x)(T) + g(x(t_1), x(t_2),\ldots,x(t_p)) = x_1$$

for the control u defined in (5), provided we can obtain a fixed point of the nonlinear operator Φ.

Let $Y = C(J, E)$ and $Y_0 = \{x : x \in Y, \|x(t)\| \le r \text{ for } t \in J\}$

where the positive constant r is given by

$$r = M_1\|x_0\| + M_1K_1 + M_1M_2M_3\left[\|x_1\| + K_1 + M_1\|x_o\| + M_1K_1 + M_1K_2T + M_1K_3T^2\right]T + M_1K_2T + M_1K_3T^2$$

Then Yo is clearly a bounded closed convex subset of Y.

Define a mapping $\Phi : Y \to Y_o$ by

$$(\Phi x)(t) = U(t)x_o - U(t)g(t_1), x(t_2),\ldots,x(t_p))$$

$$+\int_0^t U(t-\mu)BW^{-1}\left[x_1 - g(x(t_1), x(t_2),\ldots,x(t_p)) - U(T)x_o\right.$$

$$+ U(T)g(x(t_1), x(t_2),\ldots,x(t_p)) - \int_0^T U(T-s)\left\{f(s,x(s))\right.$$

$$\left.\left.+\int_0^s G(s,\tau, x(\tau), Q(\tau))d\tau\right\}ds\right](\mu)d\mu + \int_0^t U(t-s)f(s,x(s))ds$$

$$+\int_0^t U(t-s)\left[\int_0^s G(s,\tau, x(\tau), Q(\tau))d\tau\right]ds.$$

Since f, g, G and K are continuous and $\|(\Phi x)(t)\| \leq r$ it follows that Φ is also continuous and maps Y_o into itself. Moreover, Φ maps Y_o into a precompact subset of Y_o. To prove this, we first show that for every fixed t, $t \in J$, the set

$$Y_o(t) = \{(\Phi x)(t) : x \in Y_o\} \text{ is precompact in } E.$$

This is clear for $t = 0$, since $Y_o(0) = \{x_o - g\}$. Let $t > 0$ be fixed and for $0 < \varepsilon < t$, we define

$$\Phi_\varepsilon (x(t)) = U(t)x_o - U(t)g(x(t_1), x(t_2),\ldots,x(t_p))$$

$$+ \int_0^{t-\varepsilon} U(t-\mu)BW^{-1} \Big[x_1 - g(x(t_1), x(t_2),\ldots,x(t_p)) - U(T)x_o$$

$$+ U(T)g(x(t_1), x(t_2),\ldots,x(t_p))$$

$$- \int_0^T U(T-s)\Big\{ f(s,x(s)) + \int_0^s G(s,\tau, x(\tau), Q(\tau))d\tau \Big\} ds\Big](\mu)d\mu + \int_0^{t-\varepsilon} U(t-s)f(s,x(s))ds$$

$$+ \int_0^{t-\varepsilon} U(t-s) \Big[\int_0^s G(s,\tau,x(\tau), Q(\tau))d\tau\Big]ds.$$

Since U(t) is compact for every $t > 0$, the set

$$Y_\varepsilon(t) = \{(\Phi_\varepsilon x)(t) : x \in Y_o\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Furthermore, for $x \in Y_o$ we have

$$\|(\Phi x)(t) - (\Phi_\varepsilon x)(t)\|$$

$$\leq \Big\| \int_{t-\varepsilon}^{t} U(t-\tau)BW^{-1} \Big[x_1 - g(x(t_1), x(t_2),\ldots,x(t_p)) - U(T)x_o$$

$$+ U(T)g(x(t_1), x(t_2),\ldots,x(t_p)) + \int_0^T U(T-s) \{f(s,x(s))$$

$$+ \int_0^s G(s, \tau, x(\tau), Q(\tau))d\tau\}ds\Big](\mu)d\mu \,\| + \| \int_{t-\varepsilon}^t U(t-s)\,\{f(s,x(s))$$

$$+ \int_0^s G(s, \tau, x(\tau), Q(\tau))\, d\tau\}\, ds \,\|$$

$$\le \varepsilon M_1M_2M_3 \,\{\|x_1\| + K_1 + M_1\|x_o\| + M_1K_1 + M_1[K_2 + K_3T]T\}$$

$$+ \varepsilon M_1\,[K_2 + K_3T].$$

which implies that $Y_o(t)$ is totally bounded, that is precompact in E. We want to show that

$$\Phi(Y_o) = S = \{\Phi x : x \in Y_o\} \tag{7}$$

is an equicontinuous family of functions. For that let $t^* > t > 0$. Then we have

$$\| (\Phi x)(t) - (\Phi x)(t^*) \|$$

$$\le \| U(t) - U(t^*) \| \,[\, \|x_o\| + \| g(x(t_1), x(t_2),\ldots,(t_p)) \|]$$

$$+ \| \int_0^t [U(t-\mu)-U(t^*-\mu)] \,\|\, BW^{-1}\Big[x_1 - g(x(t_1),x(t_2),\ldots x(t_p)) - U(T)x_o$$

$$+ U(T)g(x(t_1), x(t_2),\ldots,x(t_p)) - \int_0^T U(T-s)\Big\{f(s,x(s))$$

$$+ \int_0^t G(s,\tau,x(\tau), Q(\tau))d\tau\Big\}ds\Big](\mu)d\mu$$

$$- \int_t^{t^*} U(t^*-\mu)BW^{-1}\Big[x_1 - g(x(t_1), x(t_2),\ldots,x(t_p)) - U(T)x_o$$

$$+ U(T)g(x(t_1), x(t_2),\ldots,x(t_p))$$

$$- \int_0^T U(T-s)\,\Big\{f(s,x(s)) + \int_0^s G(s, \tau, x(\tau), Q(\tau))d\tau\Big\}ds\Big](\mu)d\mu \,\|$$

$$+ \left\| \int_0^t [U(t-s) - U(t^*-s)] \, [f(s,x(s)) + \int_0^s G(s,\tau, x(\tau), Q(\tau))d\tau]ds \right.$$

$$\left. - \int_0^{t^*} U(t^*-s) \, [f(s,x(s)) + \int_0^s G(s,\tau,x(\tau), Q(\tau))d\tau] \right\| ds$$

$$\leq \|U(t) - U(t^*)\| \, [\|x_0\| + K_1]$$

$$+ \int_0^t \|U(t-\mu) - U(t^*-\mu)\| \, M_2M_3 \Big[\|x_1\| + K_1 + M_1 (\|x_0\| + K_1)$$

$$+ \{M_1(K_2+K_3T)\}T\Big]d\mu$$

$$+ \int_t^{t^*} \|U(t^*-\mu)\| \, M_2M_3 \Big[\|x_1\| + K_1 + M_1 (\|x_0\| + K_1) + \{M_1(K_2+K_3T)\}T\Big]d\mu$$

$$+ \int_0^t \|U(t-s) - U(t^*-s)\| \, (K_2 + K_3T)ds + \int_t^{t^*} \|U(t^*-s)\|(K_2 + K_3T)ds \tag{8}$$

The compactness of U(t), T > 0 implies that U(t) is continuous in the uniform operator topology for t > 0. Thus, the right hand side of (8), which is independent of $x \in Y_0$ and tends to zero as $t^* - t \to 0$. Thus S is equicontinuous family of functions. Also S is bounded in Y, by Arzela-Ascoli's theorem, S is precompact. Hence from the Schauder fixed point theorem that Φ has a fixed point in Y_0 (see[15]) and any fixed point of Φ is a mild solution of (1) on J satisfying $(\Phi x)(t) = x(t) \in E$. Hence the system (1) is nonlocally controllable on J.

4. EXAMPLE

In this section the state x(t,z) is interpreted as the temperature of a physical substance. Then the heat equations in the following examples represent the variation of heat for all the physical phynomena from the theory of heat conduction.

Example 4.1 Consider the heat equation

$$\frac{dx}{dt}(t,z) = x_{zz}(t,z) + (Bu)(t) + x^2{}_z(t,z) \text{ if } z \in I = (0,1)\ ; t \in J \tag{9}$$

$$x(0,t) = 0, \qquad x(1,t) = 0 \tag{10}$$

and the nonlocal boundary condition for p = 1

$$x(z,0) - x(z,1) = \chi(z), \qquad \chi(z) \in L^2(I) \tag{11}$$

and the operator $A = \frac{\partial^2}{\partial x^2}$: $H^2(0,1) \cap H_0{}^1(0,1) \to H_0{}^1(0,1)$ generates a strongly continuous compact semigroup U(t) (see [12]), x(t, z) is the internal energy. $B : V \to E$, an indentity operator with $V \subset J$ and $E = L^2(I,R)$ such that there exists an invertible operator W^{-1} on $L^2(J, U)\backslash KerW$, where W is defined by

$$Wu = \int_0^T U(T-s)u(s)ds.$$

Then the problem (9-11) can be reduced to the form (1) and by the Theorem 3.1, the system is nonlocally controllable on J.

Example 4.2 Consider the following heat equation for material with memory (see[13]) of the form

$$\frac{d}{dt}x(t,z) = x_{zz}(t,z) + (Bu)(t) + \mu_1(t, x_{zz}(t,z))$$

$$+ \int_0^t \mu_3\Big(t,s, x_{zz}(s,z), \int_0^s \mu_2(s,\tau, x_{zz}(\tau,z))d\tau\Big)ds,\ z \in I = (0,1),\ t \in J \tag{12}$$

$$x(0,t) = x(1,t) = 0 \tag{13}$$

with the nonlocal boundary condition (3) in which

$$g_i(x(t_1), x(t_2),\ldots,x(t_p)) = \sum_{k=1}^{n}\Big(\sum_{j=1}^{p} c_{kj}{}^i\, x_k(t_j)\Big),\ i = 1,2,\ldots,n. \tag{14}$$

When n = 1, this reduces to

$$g(x(t_1), x(t_2),\ldots,x(t_p)) = c_1 x(t_1) + c_2 x(t_2) + \ldots + c_p x(t_p) \tag{15}$$

Here $A = \frac{\partial^2}{\partial z^2}$; generates a strongly continuous compact semigroup U(t), x is the internal energy, Bu(t) is the heat control parameter and μ_3 is the external heat with certain heat flux. The g_i's (i = 1,2,...,n) given by (14) can be applied on the descriptions of heat propagation phenomena in which we can measure sums of the positions of a material point at the moments 0, $t_1, t_2, \ldots, t_p$ according to the following formula :

$$x_i(0) + \sum_{k=1}^{n} \left(\sum_{j=1}^{p} c_{kj}{}^{i}\, x_k(t_j)\right), \qquad i = 1,2,\ldots,n$$

$$x_i(T) + \sum_{k=1}^{n} \left(\sum_{j=1}^{p} c_{kj}{}^{i}\, x_k(t_j)\right), \qquad i = 1,2,\ldots,n$$

Here $B : V \rightarrow E$ is an indentity operator with $V \subset J$ and $E = L^2\,[I, R]$ such that there exists an invertible operator W^{-1} on $L^2\,(J, U) \setminus \ker W$, where W is defined by

$$Wu = \int_0^T U(T-s)u(s)ds,$$

$\mu_1 : J \times E \rightarrow E$, $\mu_2 : J \times J \times E \rightarrow E$ and $\mu_3 : J \times J \times E \times E \rightarrow E$ are all continuous and bounded by the positive constants. Then the problem (12 – 14) can be reduced to the form (1) by making suitable choices of A, B, f, K and G as follows (see[19]).

Let $E = L^2\,[I, R]$; $Ay = y_{zz}$

$$D(A) = \left\{y \in E \,;\, y_z, y_{zz} \in E,\, y\,\big|_{z=0,1} = 0\right\}.$$

Here $g \in E$ with $y(0) + g(y(t_1), y(t_2),\ldots,y(t_p)) = y_0$ and $B : V \rightarrow E$ are such that the condition in the hypothesis (ii) is satisfied and

$$f(t,y)(z) \;=\; \mu_1(t, s, y_{zz}(z)), \qquad (t,y) \in J \times E$$

$$K(t,s,y)(z) \;=\; \mu_2(t, s, y_{zz}(z)),$$

$$G(t,s,y,\sigma)(z) \;=\; \mu_3(t,s, y_{zz}(z), \sigma(z)), \qquad z \in I$$

Then the equations (12-14) become an abstract formulation of (1). Further all the conditions stated in the Theorem 3.1 are satisfied. Hence the system is nonlocally controllable on J.

REMARK 4.1.

The nonlocal condition (11) can be used to the description of heat effects in atomic reactors. If $x(z, 0)$ is interpreted as the given temperature in an atomic reactor at initial instant, then the atomic reactor is safest and the reaction is most dangerous according to $\chi(z)$ (for details see [5]).

REMARK 4.2.

The temperature quantities $x(t_1)$, $x(t_2)$, ..., $x(t_p)$ satisfy the nonlocal boundary condition in the general sense or in a particular sense discussed in the Example 4. 2, it is obvious to use Theorem 3. 1. and it is not necessary to know all the above quantities. It is necessary to know the relation (15) between these quantities. Therefore, the physical interpretations of nonlocal problems are significant, nonlocal problems possess deep physical meanings in this literature.

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