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WEAK CONTROLLABILITY OF SECOND ORDER SEMILINEAR VOLTERRA INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACE*

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ABSTRACT

Sufficient conditions are established for weak controllability of second order semilinear Volterra inegrodifferential equation in Banach space by using the theory of strongly continuous cosine family of linear operators. The results are obtained by using the Schauder fixed point theorem. An application to partial integrodifferential equation is given.

KEY WORDS : Controllability, fixed point theorem, Banach space, Volterra differential systems.

AMS (MOS) Subject Classification : 93B05

1. INTRODUCTION

The problem of controllability of first order dynamical systems represented by differential equations in finite dimensional space has been extensively studied using fixed point methods (see [1]). Several authors have extended the concept to infinite dimensional systems represented by evolution equations with bounded linear operators in Banach spaces (see [4,5]). Klamka [12] obtained necessary and sufficient conditions for approximate controllability of second order abstract

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differential equation by transforming the equation into the set of first order equations. Recently Kim [11] discussed boundary controllability of second order partial integrodifferential equation using a new kind of unique continuation property. Based on these recent investigations the weak controllability problem for a second order system in Banach space is formulated without converting it into a first order differential systems. The purpose of this paper is to study the controllability of higher order differential systems by using the Schauder fixed point theorem. The considered second order partial integrodifferential equation serves as an abstract formulation of many partial differential equations which arise in the problems with mathematical model of Viscoelasticiy[6], heat-flow, epidemic and other physical phenomenon (see [7,8,13]).

2. PRELIMINARIES

Consider the abstract semilinear second order Volterra integrodifferential system of the form

$$x''(t) = A\,x + \int_0^t g(t, s, x(s))ds + (Bu)(t), \; t \in J \tag{1a}$$

$$x(0) = x_0, \; x'(0) = y_0 \tag{1b}$$

where $J = [-T, T]$, $T \in R$, the state $x(t)$ takes the values in Banach space X and the control function u(.) is given in $L^2(J, U)$, a Banach space of admissible control functions, with U a Banach space. Also A is the infinitesimal generator of a strongly continuous cosine family of linear operators in X, B is a continuous bounded linear operator from U into X, and g is a continuous nonlinear mapping from J x J x X to X.

Definition 2.1 [16]. A one parameter family C(t), $t \in R$, of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if

(1) $C(s+t) + C(s-t) = 2C(s)\,C(t)$ for all $s, t \in R$,

(2) $C(0) = I$,

(3) C(t)x is continuous in t on R for each fixed $x \in X$.

If C(t), $t \in R$, is a strongly continuous consine family in X, then S(t), $t \in R$, is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)x\,ds, \; x \in X, \; t \in R. \tag{2}$$

The infinitesimal generator of a strongly continuous cosine family C(t), $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \left(d^2/dt^2\right)(C(t)x),\ t = 0,\ x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function in $t\}$. We shall also make use of the set

$$E = \{x \in X : C(t)x \text{ is a once continuously differentiable function in } t\},$$

then $S(t)X \subset E$ and $S(t)E \subset D(A)$ for $t \in R$.

Lemma 2.2 [16] If $C(t)$, $t \in R$, be a strongly continuous cosine family in X, then

1. there exist constants $L \geq 1$ and $\omega \geq 0$ so that $|C(t)| \leq Le^{\omega|t|}$ for all $t \in R$ and

$$|S(t_1) - S(t_2)| \leq L\left|\int_{t_1}^{t_2} e^{\omega|s|}\,ds\right| \text{ for all } t_1, t_2 \in R.$$

2. if $x \in E$, then $S(t)x \in D(A)$, $(d/dt)C(t)x = AS(t)x$ and $\left(d^2/dt^2\right)C(t)Ax =$

   $AC(t)x = C(t)Ax$ for $x \in D(A)$.

For the existence of solution of equation (1), assume the following conditions :

i) A is an unbounded linear operator, with bounded inverse, that generates a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X and associated sine family $S(t)$, $t \in R$, is compact with $\| S(t) \| \leq M$, $M > 0$.

ii) The fractional powers $(-A)^\alpha$ exist for $0 \leq \alpha \leq 1$ as closed linear operators in X and for $\alpha \in (0,1]$, $(-A)^\alpha$ maps $D((-A)^\alpha)$ onto X and is one-one (see [10]). Hence $D(-A)^\alpha)$ is a Banach space with the norm

$$\|x\|_\alpha = \|(-A)^\alpha x\|,\ x \in D((-A)^\alpha).$$

we denote this Banach space by X_α.

iii) The nonlinear function $g : J \times J \times D \to X$ is continuous, where D is an open subset of X_α for some $\alpha \in (0,1)$. Further, let

$$M_1 = \sup_{t,s \in J} \| g(t, s, X(s)) \|_\alpha, \text{ for } M_1 > 0.$$

iv) The linear operator W from U into X defined by

$$W u = \int_0^T S(T\text{-}s)Bu(s)ds$$

has a bounded invertible operator W^{-1} defined on $L^2(J,U)/\ker W$ and Bu is continuously differentiable. Further assume that there are positive constants N_1, N_2 such that $\| B \| \leq N_1$ and $\| W^{-1} \| \leq N_2$.

Let the assumptions (i) – (iii) hold. Then a continuous function $x : J \to X_\alpha$ which is a mild solution of equation (1) and is given by (sec [3])

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_0^s g(s,\tau,x(\tau))d\tau\, ds + \int_0^t S(t-s)Bu(s)ds \quad t\in J. \quad ...(3)$$

Definition 2.3 : The system (1) is said to be **weakly controllable** on the interval J if, for every x_0, y_0, $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution of (1) satisfies $x(T) = x_1$, where x_1 and T are preassigned terminal state and time respectively.

3. CONTROLLABILITY RESULTS

Theorem 3.1. Suppose the assumptions (i) – (iv) hold. Then for each $x \in D$ such that $x \in D(A)$ and for each $y \in E$, the system (1) is *weakly controllable* on J.

Proof : Using the above hypothesis (iv), for an arbitrary function x(.), define a control function

$$u(t) = W^{-1}\left[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-s) \int_0^s g(s,\tau,x(\tau))\, d\tau ds\right](t)$$

Also, using this control, we shall show that operator G to defined below has a fixed point. For r > 0, consider

$$N_r(x) = \{ z \in X_\alpha ; \| x - z \|_\alpha < r \}$$

Put $\phi(t) = C(t)x_0 + S(t)y_0$, then $\phi : R \to X$ is continuous. Choose r, T > 0 such that

$$N_r(x) \subset D \text{ and } \| \phi(t) - x\|_\alpha < r/2,\ t \in J.$$

Let K be the closed bounded convex subset of $C = C(J, X_\alpha)$ defined by

$$K = \{ \eta \in C : \| \eta - \phi \|_c \le r/2 \},$$

where $\| . \|_c$ denotes the supremum norm in C and

$$r/2 = MM_1T^2 + MN_1N_2T \left[\| x_1 \|_\alpha + Le^{\omega|T|} \|x_0\|_\alpha + M \|y_0 \|_\alpha + MM_1T^2\right]$$

If $\eta \in K$, then

$$\| \eta(t) - x \|_\alpha \le \| \eta(t) - \phi(t) \|_\alpha + \| \phi(t) - x \|_\alpha$$
$$\le \| \eta - \phi \|_c + \| \phi(t) - x \|_\alpha < r/2 + r/2 = r$$

So, $K \subset N_r(x) \subset D$. Define the transformation G on K by

$$(Gx)(t) = \phi(t) + \int_0^t S(t-s) \int_0^s g(s,\tau,x(\tau))d\tau ds$$

$$+ \int_0^t S(t-\theta)BW^{-1}\left[x_1 - \phi(T) - \int_0^T S(T-s) \int_0^s g(s,\tau, x(\tau))d\tau ds\right](\theta)d\theta.$$

Clearly $(Gx)(T) = x_1$, which means that the control u steers the solution of the second order Volterra integrodifferential system from the initial state x_0 to x_1 in time T provided a fixed point of the nonlinear operator Φ can be obtained with $x'(0) = y_0$.

$$\| (Gx)(t) - \phi(t) \|_\alpha \le \| (-A)^\alpha \int_0^t S(t-s) \int_0^s g(s,\tau, x(\tau))\, d\tau ds \|$$

$$+ \| \int_0^t S(t-\theta)BW^{-1}\Big[\|x_1\|_\alpha + \| C(T)(-A)^\alpha x_0 \| + \| S(T)(-A)^\alpha y_0 \|\Big]$$

$$+ (-A)^\alpha \int_0^T S(T-s) \int_0^s g(s, \tau, x(\tau))d\tau ds\Big](\theta)d\theta \|$$

$$\le MM_1T^2 + MN_1N_2T\Big[\|x_1\|_\alpha + Le^{\omega|T|} \| x_0 \|_\alpha$$

$$+ M \| y_0 \|_\alpha + MM_1T^2\Big] = r/2.$$

Thus G maps K into itself. Since ϕ, g are continuous functions, so is G (see [2]). Moreover, for $\eta_1, \eta_2 \in K$, $t \in J$,

$$\| (G\eta_1)(t) - (G\eta_2)(t) \|_\alpha$$

$$\le \| (-A)^{\alpha-1} \Big\{(-A) \int_0^t S(t-s) \int_0^s [g(s, \tau, \eta_1(\tau)) - g(s, \tau, \eta_2(\tau))]\, d\tau ds\Big\}\|$$

$$+ \| (-A)^{\alpha-1} \Big\{ (-A) \int_0^t \| S(t-\theta) \| \|BW^{-1} \| \int_0^T S(T-s)$$

$$\int_0^s [g(s, \tau, \eta_1(\tau)) - g(s, \tau, \eta_2(\tau))]\, d\tau ds \Big\}\, \theta \| d\theta,$$

and the continuity of G follows from the compactness of the operator $(-A)^{\alpha-1}$.

Next it is shown that the set $\{ Gx : x \in K \}$ is an equicontinuous family of functions in C.

$$\|(G\eta)(t_1) - (G\eta)(t_2)\|_\alpha \le \|\phi(t_1) - \phi(t_2)\|_\alpha$$

$$+ \left\| \int_0^{t_1} [S(t_1 - s) - S(t_2 - s)] \left[(-A)^\alpha \int_0^s g(s, \tau, \eta(\tau))\, d\tau\right] ds \right\|$$

$$+ \left\| \int_{t_1}^{t_2} \| S(t_2 - s) \| \left\{ (-A)^\alpha \int_0^s g(s, \tau, \eta(\tau)) d\tau \right\} ds \,\right] \Big\|$$

$$+ \left\| \int_0^{t_1} [S(t_1 - \theta) - S(t_2 - \theta)]\, BW^{-1} \right.$$

$$\left[\| x_1 \|_\alpha + \| C(T)\, (-A)^\alpha x_0 \| + \| S(T)\, (-A)^\alpha y_0 \| \right.$$

$$\left. + \int_0^T S(T-s)\, (-A)^\alpha \int_0^s g(s, \tau, \eta(\tau))\, d\tau ds \,\right] (\theta)\, d\theta \Big\|$$

$$+ \Big\| \int_{t_1}^{t_2} S(t_2 - \theta)\, BW^{-1}$$

$$\left[\| x_1 \|_\alpha + \| C(T)\, (-A)^\alpha x_0 \| + \| S(T)(-A)^\alpha y_0 \| \right.$$

$$\left. + \int_0^T S(T-s)\, (-A)^\alpha \int_0^s g(s, \tau, \eta(\tau))\, d\tau ds \right] (\theta)\, d\theta \Big\|$$

$$\le \quad \|\phi(t_1) - \phi(t_2)\|_\alpha + M_1 T \left[\int_0^{t_1} \| S(t_1 - s) - S(t_2 - s) \|\, ds \right]$$

$$+ M_1 T \left[\int_{t_1}^{t_2} \| S(t_2 - s) \|\, ds \right]$$

$$+ N_1 N_2 T \left[\| x_1 \|_\alpha + Le^{\omega|T|} \| x_0 \|_\alpha + M \| y_0 \|_\alpha + MM_1 T^2 \right]$$

$$\times \left\| \left[\int_0^{t_1} [S(t_1 - \theta) - S(t_2 - \theta)]\, d\theta \right\|\right.$$

$$+ N_1 N_2 T \left[\| x_1 \|_\alpha + Le^{\omega|T|} \| x_0 \|_\alpha + M \| y_0 \|_\alpha + MM_1 T^2 \right] \Big\|$$

$$\int_{t_1}^{t_2} S(t_2 - \theta)\, d\theta \,\|$$

$$\to 0 \text{ as } |t_1 - t_2| \to 0 \text{ uniformly for } \eta \in K.$$

Hence G(K) is an equicontinuous family of functions. Finally, we want to show that for each fixed $t \in J$, the set $\{ (Gx)(t) : x \in K \}$ is precompact in X_α (see [15]). Since $(-A)^\beta : X \to X_\alpha$ is compact for $\alpha < \beta$, it suffices to show that

$$\{(-A)^\beta [(Gx)(t) - \phi(t)] : \mu \in K\}$$

is bounded in X for $\alpha < \beta \le 1$. Then

$$\| (-A)^\beta (Gx - \phi)(t) \| \le \| (-A)^{\beta-1} [(-A) \int_0^t S(t-s)$$

$$\{ \int_0^s g(s, \tau, x(\tau))\, d\tau \} ds] \|$$

$$+ \| (-A)^{\beta-1} \{ (-A) \int_0^t S(t-\theta)\, BW^{-1} [\| x_1 \| + \| C(T) x_0 \| + \| S(T) y_0 \|$$

$$+ \| (-A)^{\beta-1} \int_0^T [(-A)\, S(T-s) \int_0^s g(s, \tau, x(\tau))\, d\tau ds] \|] \} (\theta)\, d\theta \|$$

$$\le | (-A)^{\beta-1} | MM_1T^2 + | (-A)^{\beta-1} | \{ MN_1N_2T [\| x_1 \| + Le^{\omega|t|} \| x_0 \|$$

$$+ M \| y_0 \| MM_1T^2] \}$$

Thus, G(K) is bounded in C. By Arzela-Ascoli theorem G(K) is precompact. Direct application of the Schauder fixed point theorem yields the existence of $x \in K$ such that, $(Gx)(t) = x(t)$. Since $x_0 \in D(A)$ and $y_0 \in E$, then the solution of (3) is a solution of (1). Therefore every fixed point of G is a mild solution of equation (1). Consequently, equation (1) is weakly controllable on J.

4. EXAMPLE

The second order Volterra equations appears in mathematical models of Viscoelasticity (see [6]) because it is due to the fact that under suitable assumptions on Volterra term, the equation can be converted into the following

$$w_{tt}(x, t) = w_{xx}(x, t) + f(t).$$

This equation is a wave equation that corresponds to the complete elastic case in mechanics has a weak solution with both w_x and w_t bounded and initially smooth solutions in finite time (see [8]). For the Volterra equations the finite speed of propagation is related to its weak solution w(x,t) and which is independent of the initial condition $w_t(x,t)$ (see [9,14]). This motivates to study the weak controllability of the following integrodifferential equation

$$w_{tt}(x,t) = w_{xx}(x,t) + \int_0^t \sigma(t, s, w(x,s))ds + B(u(t)),\ 0 < x < \pi,\ t \in J \quad (4)$$

$$w(0, t) = w(\pi,t) = 0,\ t \in J \quad (4a)$$

$$w(x, 0) = w_0(x) \quad (4b)$$

$$w_t(x, 0) = w_1(x),\ 0 < x < \pi. \quad (4c)$$

Let $X = L^2((0, \pi), R)$ and $B : U \to X$ with $U \subset J$, be a bounded linear operator such that Bu be continuously differentiable. Define $W : U \to X$

$$W u = \int_0^T S(T-s)Bu(s)ds$$

and there exists an bounded invertible W^{-1} defined on $L^2(J, U)/\ker W$. Also, Sine family S(t) is compact operator and $\sigma : J \times J \times X_\alpha \to X$ is continuously differentiable such that

$$|\sigma(t, s, w(x,s))| \leq M_1$$

for $M_1 > 0$ with s, $t \in J$ and $w(x,s) \in X_\alpha$. Az = z", where

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous, } z'' \in X,\ z(0) = z(\pi) = 0\}.$$

Then

$$Az = \sum_{n=1}^{\infty} -n^2 (z, z_n) z_n,\ z \in D(A),$$

where $z_n(s) = \sqrt{2} / \pi \sin ns$, n = 1,2,3, ..., is the orthonormal set of eigenvalues of A.

And so A is the infinitesimal generator of a strongly continuous cosine family C(t), (see [17]), $t \in R$, in X given by

$$C(t)z = \sum_{n=1}^{\infty} \cos(nt) (z, z_n) z_n,\ z \in X,$$

and that the associated sine family is given by

$$S(t)z = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} (z, z_n)\, z_n, \; z \in X,$$

If $\alpha = 1/2$, then A satisfies for $0 \leq \alpha \leq 1$, $(-A)^{\alpha}$ maps onto X and is 1-1, so that $D((-A)^{\alpha})$ is a Banach space when endowed with the norm

$\| x \|_{\alpha} = \| (-A)^{\alpha} x \|$, $x \in D(-A)^{\alpha}$. Denote this Banach space by X_{α}.

Further,

$$(-A)^{1/2} z = \sum_{n=1}^{\infty} n\, (z, z_n)\, z_n, \; z \in D((-A)^{1/2}),$$

$$(-A)^{-1/2}\, z = \sum_{n=1}^{\infty} (1/n)\, (z, z_n)\, z_n, \; z \in X$$

We now define the mapping $g : J \times J \times X_{1/2} \to X$ as follows $g\,(t, s, z)\,(x) = \sigma\,(t, s, z)\,(x,s))$, $z \in X_{1/2}$, $x \in [0, \pi]$. Then problem (4–4c) can be formulated abstractly as (see [17]).

$$x''(t) = A\,x + \int_0^t g(t, s, x\,(s))\, ds + (Bu)(t)$$

$$x(0) = x_0, \; x'(0) = y_0$$

Then, all the conditions stated in the above Theorem 3.1 are satisfied. So, the equation (4) is weekly controllable on J.

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