Mathematical Forum
Vol. XIII, 1999-2000

pp 16-21

ASYMPTOTIC METHOD FOR n-TH ORDER CRITICALLY DAMPED NONLINEAR SYSTEMS

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ABSTRACT

An asymptotic method has been found to obtain approximate solution of an n-th order, $n \geq 2$ critically damped weakly nonlinear systems based on the work of Krylov and Bogoliubov. The method is illustrated by an example.

AMS Subject Classification no. 34E

1. INTRODUCTION

The asymptotic method of Krylov and Bogoliubov [1] is one of the widely used techniques to study weakly nonlinear systems. The method, originally developed for obtaining periodic solution of second-order ordinary differential equation, was amplified and justified by Bogoliubov and Mitropolskii [2] and later extended to damped nonlinear systems by Popov [3]. It is perhaps noteworthy that because of the importance of physical processes involving damping, Popov's results have later been rediscovered by several authors. Murty, Deekshatulu and Krisna [4] have studied second order over-damped nonlinear systems in the sense of Krylov and Bogoliubov's method. Murty [5] has presented a unified theory to study second-order nonlinear oscillatory, damped oscillatory and non-oscillatory systems. Sattar [6] has found an asymptotic solution of a second-order critically damped nonlinear system. Sometimes, Murty, Deekshatulu and Krisna's [4] second-order solution and Sattar's [6] ... results. Author [7] has found new asymptotic ... critically damped nonlinear systems. The new solutions show good coincidence with numerical solutions for different damping forces as well as for different initial conditions including those cases, where the previous solutions obtained respectively in [4] and [6] give incorrect results.

The method has been extended to third-order nonlinear systems by Osiniskii [8], Mulholland [9] and Bojadziev [10]. Shamsul and Sattar [11] have presented a unified theory to study third-order nonlinear oscillatory, damped oscillatory and non-socillatory systems. Shamsul and Sattar [12] have also studied third-order ciritically damped nonlinear systems.

Sethna [13] has studied vibrations of dynamical systems of multiple-degree of freedom system by the same method. Pavlidis [14] has used the method to study an n-dimensional biological (oscillatory) system. In a recent paper [15], Shamsul and Sattar have found asymptotic solution of an n-dimensional over-damped nonlinear system. The aim of the present paper is to find asymptotic solution of an n-th order critically damped nonlinear differential system. The general formula is ablve to give the critically damped solution, which has been found in [7]

Let us consider a nonlinear system governmed by an n-th order ordinary differential equation

$$x^{(n)} + k_1 x^{(n-1)} + \ldots + k_n x = \varepsilon f(x, \dot{x}, \ldots, x^{(n-1)}), \tag{1}$$

where $x^{(i)}$, $i = n, n-1, \ldots$, represents i-th derivatives, $k_1, k_2, \ldots\ldots$ are constants, ε is a small parameter and f is a given nonlinear function.

The auxiliary equation of the corresponding linear equation of (1) has n roots, say $\lambda_j, j = 1,2, \ldots, n$. In the case of the cirtically damped system, the discriminant of the auxiliary equation vanishes [16], so that at least two of the roots of the auxiliary equation are equal. Sometimes more than two roots may be equal [17]. However, in this paper we have considered two roots equal, say $\lambda_{n-1} = \lambda_n$. Then the solution of the linear equation of (1) is

$$x(t, 0) = \sum_{j=1}^{n-1} a_{j,0} e^{\lambda_j t} + t\, a_{n,0} e^{\lambda_n t}, \tag{2}$$

where $a_{j,0}, j = 1, 2, \ldots, n$ are arbitrary constants.

Now we seek a solution of (1) in the form

$$x(t,\varepsilon) = \sum_{j=1}^{n-1} a_j(t) e^{\lambda_j t} + t a_n(t) e^{\lambda_n t} + \varepsilon u_1(a_1, a_2, \ldots, a_n, t)$$
$$+ \varepsilon^2 u_2(a_1, a_2, \ldots, a_n, t) + \varepsilon^3 \cdots, \tag{3}$$

where $a_j, j = 1, 2, \ldots, n$ satisfy a set of first order differential equations

$$\dot{a}_j = \varepsilon A_j(a_1, a_2, \ldots, a_n, t) + \varepsilon^2 B_j(a_1, a_2, \ldots, a_n, t) + \varepsilon^3 \cdots \tag{4}$$

Confining attention to the first few terms say m, in the series expansion of (3) and (4), we evaluate unknown functions, $u_1, u_2, \ldots$ and $A_j, B_j, j = 1,2 \ldots, n$ such that $a_j(t)$ appearing in (3) and (4) will make $x(t,\varepsilon)$ satisfy (1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a low order, usually the first. In order to determine these unknown functions, it is assumed that $u_1, u_2, \ldots$ do not contain the fundamental terms in the argument t (i.e., t^0 and t^1) of f. It is noted that the terms of the expansion of (3) at order ε^0 are known as the fundamental terms [6,7]

Now differentiating (3) n-times with respect to t, substituting for the derivatives $x^{(n)}, x^{(n-1)}, \ldots, \dot{x}$ and x in original equation (1), utilizing relations of (4) and comparing the coefficients of ε, we obtain

$$= \prod_{j=1}^{n} \left(\frac{\partial}{\partial t} - \lambda_j\right) u_1 + \sum_{j=1}^{n-1} \left(\prod_{h=1.\ k\neq j}^{n} \left(\frac{\partial}{\partial t} + \lambda_j - \lambda_k\right)\right) A_j$$

$$+ e^{\lambda_n t} \prod_{k=1}^{n-2} \left(\frac{\partial}{\partial t} + \lambda_{n-1} - \lambda_k\right) A_n$$

$$+ \frac{d}{d\lambda_n}\left(e^{\lambda_n t} \prod_{k=1}^{n-1} \left(\frac{\partial}{\partial t} + \lambda_n - \lambda_k\right) A_n \right)_{\lambda_n = \lambda_{n-1}}$$

$$= f^{(0)} (a_1, a_2, \ldots, a_n, t), \tag{5}$$

where $f^{(0)} (a_1, a_2, \ldots, a_n, t) = f(x_0, \dot{x}_0, \ldots, x_0^{(n-1)})$

$$\text{and } x_0 (t, \varepsilon) = \sum_{j=1}^{n-1} a_j(t) e^{\lambda_j t} + t a_n(t) e^{\lambda_n t}.$$

Here it is noted that when $n = 2$, the third term of (5) becomes $e^{\lambda_2 t} A_2$. According to the customary assumption of Krylov and Bogoliubov's method, (5) can be separated into $n+1$ equations for unknown functions u_1 and $A_j, j = 1,2, \ldots, n$. Therefore, we can find all unknown functions which are involved in the first order solution. The method can be carried out in higher order in the same way.

3. EXAMPLE

As an example of the above procedure, we may consider Duffing's equation with critical damping force

$$\ddot{x} + 2\dot{x} + x = -\varepsilon x^3.$$

Here, $n = 2$, $\lambda_1 = \lambda_2 = -1$, $x_0 = e^{-t}(a_1 + a_2 t)$ and

$$f^{(0)} = -e^{-3t}\left(a_1^3 + 3a_1^2 a_2 t + 3a_1 a_2^2 t^2 + a_2^3 t^3\right)$$

For nonlinear equation (6), (5) takes the form

$$\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)u_2 + e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)A_1 + e^{\lambda_2 t}A_2$$

$$+ \frac{d}{d\lambda_2}\left(e^{\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_1\right)A_2\right)_{\lambda_2 = \lambda_1} = f^{(0)}. \tag{7}$$

Now simplifying and substituting the values of λ_1, λ_2 and $f^{(0)}$ into (7) we obtain

$$\left(\frac{\partial}{\partial t} + 1\right)^2 u_1 + e^{-t}\left(\frac{\partial A_1}{\partial t} + 2A_2 + t\frac{\partial A_2}{\partial t}\right)$$

$$= -e^{-3t}\left(a_1^3 + 3a_1^2 a_2 t + 3a_1 a_2^2 t^2 + a_2^3 t^3\right). \tag{8}$$

Since u_1 does not contain fundamental term of f(0), we obtain

$$\frac{\partial A_1}{\partial t} + 2A_2 = -a_1^3 e^{-2t}, \tag{9}$$

$$\frac{\partial A_2}{\partial t} = -3a_1^2 a_2 e^{-2t}, \tag{10}$$

and

$$\left(\frac{\partial}{\partial t} + 1\right)^2 u_1 = -e^{-3t}\left(3a_1 a_2^2 t^2 + a_2^3 t^3\right) \tag{11}$$

The particular solution of (9)-(11) are

$$A_1 = \frac{e^{-2t}\left(a_1^3 + 3a_1^2 a_2\right)}{2}$$

$$A_2 = \frac{3e^{-2t}a_1^2 a_2}{2} \tag{12}$$

and

$$u_1 = -\frac{e^{-3t}}{4} \times$$

$$\left(\frac{9a_1 a_2^2}{2} + 3a_2^3 + \left(6a_1 a_2^2 + \frac{9a_2^3}{2}\right)t + 3\left(a_1 a_2^2 + a_2^3\right)t^2 + a_2^3 t^3\right) \tag{13}$$

Since ε is small, we may replace a_1 and a_2 by their corresponding linear values i.e., $a_{1,0}$ and $a_{2,0}$ in the right hand sides of (12) [18]. Thus substituting thje values of A_1 and A_2 from (12) into (4) and then integrating with respect to t, we obtain

$$a_1 = a_{1,0} + \frac{\varepsilon\left((a_{1.0}^3 + 3a_{1.0}^2 a_{2,0})\right)\left(1-e^{-2t}\right)}{4},$$

$$a_2 = a_{2,0} + \frac{3\varepsilon a_{1.0}^2 a_{2.0}\left(1 - e^{-2t}\right)}{4}, \qquad (14)$$

where we have also considered $a_1(0) = a_{1,0}$ and $a_2(0) = a_{2,0}$ in accordance to [18].

Hence the first order solution of (6) is

$$x = e^{-t}(a_1 + a_2 t) + \varepsilon u_1, \qquad (15)$$

where u_1 is given by (13) and a_1 and a_2 are given by (14). Solution (15) is similar to that solution obtained in [7].

4. CONCLUSION

A general formula for critically damped nonlinear systems has been found. The method is a generalization of Krylov and Bogoliubov's [1] asymptotic method and it has been applied to a second-order nonlinear system in Sec. 3. In [7], it has been shown that solution (15) shows a good agreement with those obtained by numerical method for different initial conditions. So, we are not interested to repeat it. Moreover, it has been shown in [7] that solution (15) can be brought to Sattar's solution by suitable substitution. But it has also been shown in [7] that Sattar's solution is useful for certain initial conditions only. The method can be applied to third-order or higher order weakly nonlinear system in the same way.

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