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APPROXIMATE SOLUTIONS OF NON-OSCILLATORY SYSTEMS

M. SHAMSUL ALAM
Department of Mathematics
Bangladesh Institute of Technology,
Rajshahi - 6204, Bangladesh

ABSTRACT :

Krylov-Bogoliubov-Mitropolskii method has been extended to certain non-oscillatory nonlinear systgems. The solutions obtained for different initial conditions for a second-order nonlinear system whose unperturbed equations has two real and non-positive characteristic roots show a good coincidence with those obtained by numerical method.

1. Introduction

Krylov-Bogoliubov [1] initiated a perturbation method to obtain approximate solution (oscillatory-type) of the second-order nonlinear differential equation with a small non-linearity

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \tag{1}$$

where the over-dots denote differentiation with respect to t, $\omega_0 > 0$ and ε is a small parameter. Then the method was amplified and justified by Bogoliubov and Mitropolskii [2,3]. Today the method is a well-known method as Krylov-Bogoliubov-Mitropolskii (KBM) [1,2] method in the literature of nonlinear oscillations. Popov [4] extended the method to the following damped oscillatory system

$$\ddot{x} + c\,\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}), \tag{2}$$

where $c > 0$, $\omega > 0$ and $c < 2\omega$. It is to be noted that if $c \geq 2\omega$, the system (2) becomes non-oscillatory. First, Murty *et al* [5,6] used this method to obtain approximate solution of (2) characterized by non-oscillatory processes. They actually found the over-damped solutions of (2). In the case of over-damped systems, we know the inequality $c > 2\omega > 0$ and the characteristic roots of the unperturbed equation of (2) become real, unequal and negative. It is also to be noted that the roots of the unperturbed equation of (1) are purely imaginary. On the contrary, the roots of (2) are complex conjugate with negativbe real part when $c < 2\omega$ (considered by Popov [4]). Sattar [7] found an approximate solution of (2) characterized by critical damping. However, in the case of a critically damped system, we get the equality $c = 2\omega$ and the roots of (2) become real, equal and negative. Author [8] studied a nonlinear system described by (2) in which two roots and real, almost equal (rather than equal) and negative. However, a simple and interesting case of (2) in which ω is small or $\omega \to 0$ has remained almost untouched. The aim of this paper is in part to fill that gap and to provde a basis for further generalization.

2. Method

Consider the second-order nonlinear differential equation

$$\ddot{x} + c\dot{x} = -\varepsilon f(x,\dot{x}), \tag{3}$$

where $c > 0$ and ε is again a small parameter. It is obvious that the third linear term of (2) has either contained in f of (3) or it vanishes. The aim of our present study is to obtain an approximate solution of (3). The characteristic roots of the unperturbed equation of (3) are 0, $-c$; so that the solution of the unperturbed equation of (3) becomes

$$x(t,0) = a_0 + b_0 e^{-ct}, \tag{4}$$

where a_0 and b_0 are arbitrary constants.

We choose an approximate solution of (3) followed by KBM method (see section 6 for details, in particular equation (19)) in the form

$$x(t,\varepsilon) = a(t) + b(t)e^{-ct} + \varepsilon\mu_1 (a,b,t) + \varepsilon^2 \ldots, \tag{5}$$

where a and b satisfy the differential equations

$$\dot{a} = \varepsilon A_1 (a, b, t) + \varepsilon^2 \ldots,$$

$$\dot{b} = \varepsilon B_1 (a, b, t) + \varepsilon^2 \ldots, \tag{6}$$

Now differentiating (5) twice with respect to t and utilizing relations of (6), we obtain

$$\dot{x} = -cbe^{-ct} + \varepsilon\left(A_1 + e^{-ct}B_1 + \frac{\partial u_1}{\partial t}\right) + \varepsilon^2 \ldots,$$

$$\ddot{x} = c^2 be^{-ct} + \varepsilon\left(\frac{\partial A_1}{\partial t} + e^{-ct}\left(\frac{\partial}{\partial t} - 2c\right) B_1 + \frac{\partial^2 u_1}{\partial t^2}\right) + \varepsilon^2 \ldots, \tag{7}$$

Substituting the values of $\dot{x}$, $\ddot{x}$ from (7) and x from (5) in equation (3), and comparing the coefficients of ε, we obtain.

$$\left(\frac{\partial}{\partial t} + c\right) A_1 + e^{-ct}\left(\frac{\partial}{\partial t} - c\right)B_1 + \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + c\right)u_1 = -f^{(0)} (a, b, t), \tag{8}$$

where $f^{(0)}(a,b,t) = f(x_0, \dot{x}_0)$ and $x_0 = a(t) + b(t)e^{-ct}$.

In general, $f^{(0)}$ can be expanded in Taylor's series as

$$f^{(0)} = \sum_{r=0}^{\infty} F_r(a,b)e^{-rct}. \tag{9}$$

It is customary in KBM method that u_1 does not cntain fundamental terms, i.e. the terms with e^{-0ct} and e^{-1ct}. Substituting the functional value of $f^{(0)}$ from (9) into (8) and equating the coefficients of e^{-0ct}, e^{-1ct}, $r \geq 2$, we obtain

$$\left(\frac{\partial}{\partial t} + c\right) A_1 = -F_0, \tag{10}$$

$$\left(\frac{\partial}{\partial t} - c\right) B_1 = -F_1, \tag{11}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + c\right)u_1 = -\sum_{r=2}^{\infty} F_r e^{-rct}. \tag{12}$$

The particular solution of (10)–(12) gives the unknown functions A_1, B_1 and u_1, which complete the determination of the first order solution of (3). The method can be carried out to higher orders in a similar way.

3. Example

As an example of the above procedure, we may consider the equation

$$\ddot{x} + 2\dot{x} = -\varepsilon x^3. \tag{13}$$

Here, $c = 2$ and the function $f^{(0)}$ becomes $f^{(0)} = a^3 + 3a^2be^{-2t} + 3ab^2e^{-4t} + b^3e^{-6t}$. The nonzero coefficients of $f^{(0)}$ are $F_0 = a^3$, $F_1 = 3a^2b$, $F_2 = 3ab^2$ and $F_3 = b^3$. Putting these values of $F_0, \ldots, F_3$ into (10)-(12) and then solving them, we obtain

$$A_1 = -\frac{1}{2}a^3, \quad B_1 = \frac{3}{2}a^2b, \tag{14}$$

and

$$u_1 = -\frac{3}{8}ab^2e^{-4t} - \frac{1}{24}b^3e^{-6t}. \tag{15}$$

Substituting the values of A_1 and B_1 from (14) into (6) and then integrating with respect to t, we obtain

$$a = a_0\left(1 + \varepsilon a_0^2 t\right)^{-\frac{1}{2}}, \quad b = b_0\left(1 + \varepsilon a_0^2 t\right)^{\frac{3}{2}} \tag{16}$$

Therefore, the first order solution of (13) is

$$x(t,\varepsilon) = a + be^{-2t} + \varepsilon u_1, \tag{17}$$

where a and b are given by (16) and u_1 is given by (15).

4. Initial Conditions

The initial conditions as obtained from (17), (7), (14) and (15) are

$$x(0,\varepsilon) = a_0 + b_0 - \varepsilon\left(\tfrac{3}{8}\, a_0 b_0^2 + \tfrac{1}{24}\, b_0^3\right),$$

$$\dot{x}(0,\varepsilon) = -2b_0 + \varepsilon\left(-\tfrac{1}{2}a_0^3 + \tfrac{3}{2}a_0^2 b_0 + \frac{3}{2}a_0 b_0^2 + \frac{1}{4}\, b_0^3\right) \tag{18}$$

Usually, in a problem the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equation in order to determine two arbitrary constants a_0 and b_0 that appear in the solution, from the initial condition equation (18). In general, (18) are solved by *Newton-Raphson* formula.

5. Results and Discussions

For every positive values of ε, the first term of perturbation solution (17) decreases very slowly with increasing t, while thesecond term (together with first order correction term, εu_1) dies out quickly, so that $x(t,\varepsilon) \cong a = a_0 \left(1 + \varepsilon a_0^2 t\right)^{-\frac{1}{2}}$, $t > 1$. The solution (17) behaves as an asymptotic solution and $x(t,\varepsilon)$ vanishes as the limit $t \to \infty$.

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to the work of Murty and Deeksatulu [5] who first found an over-damped solution of a second order nonliner system. In the present paper, for different initial conditions we have compared the perturbation solution (17) to those obtained by Runge-Kutta (fourth-order) method.

First of all, $x(\varepsilon,t)$ has been computed by perturbation solution (17) with initial conditions $[x(0) = 1, \dot{x}(0) = -1]$ or $a_0 = 0.487602$, $b_0 = 0.517881$ for $\varepsilon =$ 0.1. Corresponding numerical solution has been obtained and th epercentage errors have been calculated. All the results have been shown in **Table 1.** From **Table 1** it is seen that the errors of the results obtained by (17) are less than 1% (since $\varepsilon = 0.1$, error(s) should be occurred in an order of 1%). Then $x(\varepsilon,t)$ has been computed by (17) and numerical method with initial conditions $[x(0) = 0.5, \dot{x}(0) = 0]$ or $a_0 = 0.503248$, $b_0 = -0.003248$ for similar value of ε. The errors are again less than 1% (see **Table 2**). Comparing the values of x and a in **Tables 1** and 2, we see that they are almost equal for all values of $t \geq 5$.

Table 1

t	a	x	x_{nu}	$E(\%)$
0.0	1.000000	1.000000	1.000000	0.0000
1.0	0.481907	0.554412	0.554381	0.0056
3.0	0.471091	0.472514	0.472274	0.0508
5.0	0.460972	0.460999	0.460607	0.0851
10.0	0.438276	0.438276	0.437609	0.1524
15.0	0.418634	0.418634	0.417792	0.2015
20.0	0.401416	0.401416	0.400465	0.2315
25.0	0.386161	0.386161	0.385144	0.2641
30.0	0.372523	0.372523	0.371468	0.2840
35.0	0.360235	0.360235	0.359161	0.2990
40.0	0.349088	0.349088	0.348009	0.3100
45.0	0.338915	0.338915	0.337840	0.3182
50.0	0.329583	0.329583	0.328519	0.3239

Table 2

t	a	x	x_{nu}	$E(\%)$
0.0	0.5000000	0.500000	0.500000	0.0000
1.0	0.496994	0.496538	0.496471	0.0135
3.0	0.485155	0.385146	0.484876	0.0557
5.0	0.474124	0.474123	0.473679	0.0937
10.0	0.449533	0.449533	0.448777	0.1685
15.0	0.428411	0.428411	0.427464	0.2215
20.0	0.410011	0.410011	0.408948	0.2599
25.0	0.393794	0.393794	0.392664	0.2878
30.0	0.379362	0.379362	0.378195	0.3086
35.0	0.366408	0.366408	0.365227	0.3234
40.0	0.354696	0.354696	0.353515	0.3341
45.0	0.344040	0.344040	0.342868	0.3418
50.0	0.334291	0.334291	0.333133	0.3476

x_{nu} is computed by Runge-Kutta (fourth-order) method

6. Previous Solutions obtained by KBM method

6.1 Krylov-Bogoliubov-Mitropolskii [1,2] solution

In accordance to Krylov, Bogoliubov and Mitropolskii's [1,2] investigation, (1) has an approximate solution of the form

$$x(t,\varepsilon) = a(t)e^{i\omega_0 t} + b(t)e^{-i\omega_0 t} + \varepsilon u_1(a,b,t) + \varepsilon^2 \ldots, \tag{19}$$

where $\pm i\omega_0$ are two roots of the unperturbed equation of (1) and a, b satisfy (6).

Differentiating (19) twice with respective to t, substituting for the derivatives $\dot{x}$, $\ddot{x}$ and x in (1), utilizing relations (6) and comparing the coefficients of ε (discussed in **Sec. 2**), one obtains

$$e^{i\omega_0 t}\left(\frac{\partial}{\partial t} + 2i\omega_0\right)A_1 + e^{-i\omega_0 t}\left(\frac{\partial}{\partial t} - 2i\omega_0\right)B_1 + \left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)u_1 = -f^{(0)}(a,b,t),$$

where $f^{(0)}(a,b,t) = f(x_0, \dot{x}_0)$ and $x_0 = a(t)e^{i\omega_0 t} + b(t)e^{-i\omega_0 t}$.

In order to determine the unknown functions A_1, B_1 and u_1 form (20), it was early imposed by Krylov, Bogoliubov and Mitropolskii's [1,2] that u_1 excluded the terms $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$, since these are already included in the series expansion of (19) as the leading terms. Moreover, They assumed that $f^{(0)}$ be expanded in the Fourier series

$$f^{(0)} = \sum_{n=-\infty}^{\infty} F_n(a,b)e^{ni\omega_0 t}. \tag{21}$$

Substituting $f^{(0)}$ from (21) into (20) and assuming that u_1 excludes the terms $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$, one obtains

$$\left(\frac{\partial}{\partial t} + 2i\omega_0\right)A_1 = -F_1, \tag{22}$$

$$\left(\frac{\partial}{\partial t} - 2i\omega_0\right)B_1 = -F_{-1}, \tag{23}$$

and

$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)u_1 = -\sum_{n=-\infty}^{\infty}{}' F_n(a,b)e^{ni\omega_0 t}, \tag{24}$$

where $\sum'$ excludes the terms $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$.

When $f = x^3$ or $f^{(0)} = a^3e^{3i\omega_0 t} + 3a^2be^{i\omega_0 t} + 3ab^2e^{-i\omega_0 t} + b^3e^{-3i\omega_0 t}$, (22)-(24) become

$$\left(\frac{\partial}{\partial t} + 2i\omega_0\right)A_1 = -3a^2b, \tag{25}$$

$$\left(\frac{\partial}{\partial t} - 2i\omega_0\right)B_1 = -3ab^2, \tag{26}$$

and
$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right) u_1 = -a^3 e^{3i\omega_0 t} - b^3 e^{-3i\omega_0 t}. \tag{27}$$

Solving (25)-(27), one obtains

$$A_1 = -\frac{3a^2 b}{2i\omega_0}, \quad B_1 = \frac{3ab^2}{2i\omega_0}, \tag{28}$$

and
$$u_1 = \frac{a^3 e^{3i\omega_0 t} + b^3 e^{-3i\omega_0 t}}{8\omega_0^2} \tag{29}$$

Substituting the values of A_1 and B_1 form (28), into (6) yields

$$\dot{a} = -\frac{3\varepsilon a^2 b}{2i\omega_0}, \quad \dot{b} = -\frac{3\varepsilon ab^2}{2i\omega_0}. \tag{30}$$

Thus when $f = x^3$, the first order approximate solution of (1) is

$$x(t,\varepsilon) = ae^{i\omega_0 t} + be^{-i\omega_0 t} + \varepsilon u_1, \tag{31}$$

where a and b are solution of (30) and u_1 is given by (29). Solution (31) is in a complex form. However, by the transformations $a = \frac{1}{2}\alpha e^{i\varphi}$, $b = \frac{1}{2}\alpha e^{-i\varphi}$, (31) together with (30) and (29) can be brought to the real form [9] of KBM [1,2] solution. It notes that a and f are respecrtively the time dependent amplitude and phase.

6.2 Murty and deekshatulu's [5] solution

Murty and Deekshatulu [5] found an approximate solution of (2) in the form

$$x(t,\varepsilon) = a(t)e^{-\lambda_1 t} + b(t)e^{-\lambda_2 t} + \varepsilon u_1(a,b,t) + \varepsilon^2 \ldots, \tag{32}$$

where $-\lambda_1$ and $-\lambda_2$ two non-vanishing real and unequal roots of the unperturbed equation of (2) and a, b again stisfy (6).

Differentiating (32) twice with respective to t, substituting for the derivatives $\dot{x}$, $\ddot{x}$ and x in (2), utilizing relations (6) and comparing the coefficients of ε, one obtains

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right) A_1 + e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right) B_1 + \left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right) u_1 = -f^{(0)}(a,b,t), \tag{33}$$

where $f^{(0)}(a,b,t) = f(x_0, \dot{x})$ and $x_0 = a(t)e^{-\lambda_1 t} + b(t)e^{-\lambda_2 t}$.

In order to determine the unknown functions A_1, B_1 and u_1 form (33), Murty and Deekshatulu [5] assumed thst u_1 excluded the terms $e^{-\lambda_1 t}$ and $e^{-\lambda_2 t}$. They also asumed that $f^{(0)}$ be expanded in the taylor's series

$$f^{(0)} = \sum_{r=1}^{\infty} [h_r(a, b, t)e^{-r\lambda_1 t} + g_r(a, b, t)e^{-r\lambda_2 t}]. \tag{34}$$

Substituting $f^{(0)}$ from (34) into (33) and assuming that u_1 excludes terms $e^{-\lambda_1 t}$ and $e^{-\lambda_2 t}$, one obtains

$$\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right) A_1 = -h_1, \tag{35}$$

$$\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right) B_1 = -g_1, \tag{36}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right)u_1 = -\sum_{r=1}^{\infty} [h_r(a,b,t)e^{-r\lambda_1 t} + g_r(a,b,t)e^{-r\lambda_2 t}] \tag{37}$$

When $f = x^3$ or $f^{(0)} = a^3e^{-3\lambda_1 t} + 3a^2be^{-(2\lambda_1+\lambda_2)t} + 3ab^2e^{-(\lambda_1+2\lambda_2)t} + b^3e^{-3\lambda_2 t}$, (35)-(37) become

$$\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right) A_1 = -3ab^2e^{-2\lambda_2 t}, \tag{38}$$

$$\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right) B_1 = -3a^2be^{-2\lambda_1 t}, \tag{39}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right)u_1 = -(a^3e^{-3\lambda_1 t} + b^3e^{-3\lambda_2 t}). \tag{40}$$

Solving (38)-(40), one obtains

$$A_1 = \frac{3ab^2e^{-2\lambda_2 t}}{\lambda_1+\lambda_2},\ B_1 = \frac{3a^2be^{-2\lambda_1 t}}{\lambda_1 + \lambda_2}, \tag{41}$$

and

$$u_1 = -\frac{a^3e^{-3\lambda_1 t}}{2\lambda_1(3\lambda_1 - \lambda_2)} + \frac{b^3e^{-3\lambda_2 t}}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{42}$$

Substituting the values of A_1 and B_1 form (41), into (6) yields

$$\dot{a} = \frac{3\varepsilon ab^2e^{-2\lambda_2 t}}{\lambda_1 + \lambda_2},\ \dot{b} = \frac{3\varepsilon a^2be^{-2\lambda_1 t}}{\lambda_1 + \lambda_2} \tag{43}$$

Therefore, when $f = x^3$, the first order over-damped solution of (2) (obtained in [5] is

$$x(t,\varepsilon) = a(t)e^{-\lambda_1 t} + b(t)e^{-\lambda_2 t} + \varepsilon u_1, \tag{44}$$

where a and b are solution of (43) and u_1 is given by (42). It is obvious that solution (44) is not useful when $3\lambda_1 = \lambda_2$ or $\lambda_1 = 3\lambda_2$, since one of the denominators in the expression of u_1 vanishes. On the other hand, when $\lambda_1 = 0$ or $\lambda_2 = 0$ one of the denominators in the expression of u_1 again vanishes. Thus Murty and Deekshatulu's [5] solution is not useful whether the roots are in some integral ratio or one of the roots vanishes. Similarly, one is able to show that Murty, Deekshatulu and Krishna's [6] solution is not useful for the case mentioned above, since u_1 is same in both solutions obtained by Murty *et al* [5,6] while the solutions of a and b are different. Author [10] has recently found over-damped solutions of (2) where the roots are in integral ratio.

7. Conclusion

A simple perturbation solution of a non-oscillatory nonlinear system has been found by KBM method. The perturbation solution decreases slowly and vanishes as the limit $t \to \propto$, while the unperturbed solution approaches toward a constant value, namely a_0 as the same limt. The solution shows a good coincidence with that obtained by numerical method. The method has an another merit that when t is much greater than 1, a is alomost equal to x ; so that one finds the solution of (13) by evaluating only the term a from (16) instead of computing x from (17) together with (16), (15).

The previous non-oscillatory solutions obtained by Murty *et al* [5.6] are not useful when one of the characteristic roots of the unperturbed equation vanishes. In this case, the approximate solution (17) has filled that gap. Moreover, it should be mentioned that the solution (31) of (1) found by Krylov, Bogoliubov and Mitropolskii [1,2] is a limiting case (for $c \to 0$) of the general solution of (2) found by Popov [4] while (17) is an another limiting case of (2) as $\omega \to 0$.

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