Mathematical Forum
Vol. XIV, 2001-2002 pp 27-32

ON GENERALIZATION OF BANACH'S FIXED POINT THEOREM IN UNIFORM SPACES

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ABSTRACT :

The purpose of this paper is to prove some common fixed point theorems for weak-compatible maps of type (A) which generalize the Banach's fixed point theorem in Uniform space by taking a control function ϕ.

Key words : Uniform sapces, weak compatible maps of type (A), fixed point.

AMS Subject classification : Primary 54H25, Secondary 47H10.

1. Introduction : In a complete metric space X Banach's classical fixed point theorem guarantees the existance of a unique fixed point for a self map T of X if it is satisfies,

$$d\,(Tx, Ty) \leq k\, d\,(x, y),\ 0 \leq k \leq 1,\ x, y \in X\text{''}.$$

Browder [2] generalized the above theorem by taking a control function ϕ while Jungck [3] extended the same theorem for two commuting maps in a complete metric space.

An equivalent to the concepts of compatible and compatible maps of type (A), under some conditions Pathak-Kang-Beak [8, 9] introduced the concept of weak compatible maps of type (A) in Menger and 2-metric spaces. This concept is more general than that of weak commutativity studied by Sessa [11]. Compatibility and Compatibility of type (A) of maps was first introduced by Jungck [4] and Jungck-Murthy-Cho [5] respectively.

In this paper we deduce definition of weak compatible maps of type (A), which analogous to in [8, 9] and use it to generalize the Banach's fixed point theorem in Uniform space for four maps by taking a control function ϕ as under.

2. *Preliminaries :* A uniform space is a generalization of a metric space. Throughout the discussion (X, U) stands for Hausdorff Uniform space. For the terminology and basic properties of Uniform spaces the reader is refered to Acharya [1], Sharma [12], Mishra [7] and Rhoades [10].

Definition 2.1 : Let (X, U) be a Hausdorff Uniform space and P be a fixed family of pseudometrics p on X which generates the uniformity U. Following Kelley ([6] chapter 6), we define

(a) $V_{(p,r)} = \{ (x, y) : x, y \in X, p(x,y) < r, r > 0 \}$

(b) $G = \{ V : V = \bigcap_{i=1}^{n} V_{(p_i, r_i)} ; p_i \in P, r_i > 0, i = 1,2, \ldots, n \}$.

(c) For $\alpha > 0$,

$$\alpha V = \{ \bigcap_{i=1}^{n} V_{(p_i, r_i)} ; p_i \in P, r_i > 0, i = 1,2, \ldots, n \}.$$

Definition 2.2 : Two self maps S and T of X are said to be weak-compatible of type (A) if

$\lim_{n\to\infty} p(STx_n, TTx_n) \le \lim_{n\to\infty} p(TSx_n, TTx_n)$,

and $\lim_{n\to\infty} p(TSx_n, SSx_n) \le \lim_{n\to\infty} p(STx_n, SSx_n)$,

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = y$ for some y in X.

3. *Common fixed point theorems :* Before giving our main results we mention the following lemmas which are required in the sequal.

Lemma 3.1 [1] : Let p be any pseudometric on X and $\alpha, \beta > 0$. If $(x, y) \in \alpha V_{(p,r_1)} \circ \alpha V_{(p,r_2)}$, then $p(x,y) \le \alpha r_1 + \alpha r_2$.

Lemma 3.2 [1] : Let V be any member of G, then there is a pseudometric p on X such tht $V = V_{(p,1)}$.

This p is called Minkowski's pseudometric of V.

Lemma 3.3 : Let A, B, S and T be self mappings of X satisfying :

(3.3.1) $A(X) \subseteq T(X)$; $B(X) \subseteq S(X)$.

(3.3.2) For any $V \in G$, $\alpha > 0$; $x, y \in X$, $(Sx, Ty) \in \alpha V$ implies $(Ax, By) \in \phi(\alpha) V$ where $\phi : [0, \infty) \to [0, \infty)$ is non-decreasing continuous on the right and for $k > 0$, $\sum_{1}^{\infty} \phi^n(k) < \infty$.

(3.3.3) Let $x_0 \in X$ be arbitrary, then in virtue of (3.3.1) there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$ and so on. Inductively we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$; n = 0, 1, 2, 3, ...

Then the sequence $\{y_n\}$ is a Cauchy sequence in X.

Proof : Let $V \in G$ be arbitrary p be Minkowski's pseudometric of V. For $x, y \in X$, set $p(Sx, Ty) = r$. For $\varepsilon > 0$, we have $p(Sx, Ty) \in (r+\varepsilon)V$.

From (3.3.2), $(Ax, By) \in \phi(r+\varepsilon)V$. Using lemma 3.1 and 3.2, we get

$$p(Ax, By) < \phi(r+\varepsilon).$$

Since ε is arbitrary, we have

(3.3.4) $$p(Ax, By) \leq \phi(p(Sx, Ty)).$$

Now for sequence $\{y_n\}$ defined in (3.3.3). Using (3.3.4), we have $p(y_{2n}, y_{2n+1}) \leq \phi(p(y_{2n-1}, y_{2n}))$ and $p(y_{2n+1}, y_{2n+2}) \leq \phi(p(y_{2n}, y_{2n+1}))$.

In general,

$$p(y_n, y_{n+1}) \leq \phi(p(y_{n-1}, y_n)) \leq \phi^2(p(y_{n-2}, y_{n-1})) \leq \ldots \leq \phi^n(p(y_0, y_1)).$$

Now for any two positive integer n, m(>n), we have

$$\begin{aligned} p(y_n, y_m) &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \ldots + p(y_{m-1}, y_m), \\ &\leq (\phi^n + \phi^{n+1} + \ldots + \phi^{m-1})\, p(y_0, y_1), \\ &= \sum_{l=1}^{m-1} \phi^l (p(y_0, y_1)). \end{aligned}$$

Since for $k > 0$, $\sum_{1}^{\infty} \phi^n(k) < \infty$, we can find a positive integer n_0 such that for $m > n \geq n_0$,

$\sum_{l=1}^{m-1} \phi^l(p(y_0, y_1)) < 1$. Therefore $(y_n, y_m) \in V$ when $m > n \geq n_0$.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Lemma 3.4 : Let A, B, S and T be self maps of X satisfying (3.3.2), (3.3.3) and

(3.4.1) $$A(X) \cup B(X) \subseteq S(X) \cap T(X).$$

Then the conclusion of lemma 3.3 holds.

Proof : Since in vertue of (3.4.1) we can define a sequence $\{y_n\}$ in X as in (3.3.3) and the proof is same as the proof of lemma 3.3.

Lemma 3.5 : Let A, B, S and T be self maps of X satisfying (3.3.2), (3.3.3), (3.4.1) and

(3.5.1) $S(X) \cap T(X)$ is sequentially complete subspace of X.

Then (A, S) and (B, T) have coincidence points in X.

Proof : From lemma 3.4, $\{y_n\}$ is a Cauchy sequence in $S(X) \cap T(X)$. By completeness of $S(X) \cap T(X)$, $\{y_n\}$, consequently the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n+2}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also, converges to some z in $S(X) \cap T(X)$. Hence there exists points u, v in X such that Su = z and Tv = z.

Using (3.3.4), we have

$$p(Au, Bx_{2n+1}) \leq \phi(p(Su, Tx_{2n+1})),$$

letting $n \to \infty$, we get $p(Au, z) \leq \phi(p(z,z))$. Since for $k > 0$, $\sum_{1}^{\infty} \phi^n(k) < \infty$, it follows that $\phi(k) < k$. Therefore $p(Au, z) < p(z,z)$ yields $Au = z = Su$. Similarly,

$$P(Ax_{2n}, Bv) \leq \phi(p(Sx_{2n}, Tv)),$$

letting $n \to \infty$, we get $p(z, Bv) \leq \phi(p(z,z)$ yields $Bv = z = Tv$.

Lemma 3.6 : Let S and T be self, weak compatible maps of type (A) of X. If Su = Tu for some u in X then STu = SSu = TTu = TSu.

Proof : Let $\{x_n\}$ be a sequence in X defined by $x_n = u$; n = 1, 2, 3, ... Then we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Su$. By weak compatibility of type (A), we have $p(STu, TSu) = \lim_{n\to\infty} p(STx_n, TTx_n) \leq \lim_{n\to\infty} p(TSx_n, TTx_n) = 0$, which implies that STu = TSu. Hence STu = SSu = TTu = TSu.

Theorem 3.1 : Let A, B, S and T be self mappings of sequentially complete X satisfying (3.3.1), (3.3.2), (3.3.3) and

(3.1.1) One of A, B, S and T is continuous.

(3.1.2) (A, S) and (B,T) are the pairs of weak compatible maps of type (A).

Then A, B, S and T have a unique common fixed point in X.

Proof : From lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X. By completeness of X, $\{y_n\}$, consequently the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also, converges to some z in X.

Let S be continuous then $SAx_{2n}, SSx_{2n} \to Sz$ and the pair (A, S) is weak compatible of type (A), we have

$\lim_{n\to\infty} p(ASx_{2n}, SSx_{2n}) \leq \lim_{n\to\infty} p(SAx_{2n}, SSx_{2n}) = 0$ yields $ASx_{2n} \to Sz$.

Using (3.3.4), we have

$$p(ASx_{2n}, Bx_{2n+1}) \leq \phi(p(SSx_{2n}, Tx_{2n+1})),$$

letting $n \to \infty$, we get $p(Sz, z) \leq \phi(p(Sz, z)) < p(Sz, z)$ yields $Sz = z$.

Further, $p(Az, Bx_{2n+1}) \leq \phi(p(Sz, Tx_{2n+1}))$,

letting $n \to \infty$, we get $p(Az, z) \leq \phi(p(z,z)) < p(z,z)$ yields $Az = z = Sz$.

Now since $A(X) \subseteq T(X)$, there exists a point u in X such that $z = Az = Tu$.

Using (3.3.4), we have

$p(z, Bu) = p(Az, Bu) \leq \phi(p(Sz, Tu)) = \phi(p(z,z))$ yields $z = Bu$.

Therefore $Bu=Tu$ and the pair (B, T) is weak compatible maps of type (A) then from lemma 3.6, $BTu = BBu = TTu = TBu$, i.e. $Bz = Tz$.

Wsing (3.3.4), we have

$p(Z, Bz) = p(Az, Bz) \leq \phi(p(Sz, Tz) = \phi(p(z,Bz)) < p(z, Bz)$ yields $z = Bz$.

Thus $Az = Bz = Sz = Tz = z$ i.e. z is the common fixed point of A, B, S and T.

Now for uniqueness of z, let z_1 be another common fixed point of A, B, S and T then from (3.3.4), we have

$p(z, z_1) = p(Az, Bz_1) \leq \phi(p(Sz, Tz_1)) = \phi(p(z,z_1)) < p(z,z_1)$ yields $z = z_1$.

This completes the proof.

Without making use of continuity of maps we prove our next result.

Theorem 3.2 : Let A, B, S and T be self maps of X satisfying (3.3.2), (3.3.3), (3.4.1) (3.5.1) and (3.1.2).

Then A, B, S and T have a unique common fixed point in X.

Proof : Using conditions (3.3.2), (3.3.3), (3.4.1) and (3.5.1), from lemma 3.4-3.6, there exists point z, u,v in X such that

$z = Au = Su$, $z = Bv = Tv$ and

$SAu = SSu = AAu = ASu$, $BTv = BBv = TTv = TBv$ or $Sz = Az$, $Bz = Tz$.

Using (3.3.4), we have

$$p(Az, Bx_{2n+1}) \leq \phi(p(Sz, Tx_{2n+1})),$$

letting $n \to \infty$, we get

$p(Az, z) \leq \phi(p(Az, z)) < p(Az, z)$ yields $z = Az = Sz$.

Similarly, $p(Ax_{2n}, Bz) \leq \phi(p(Sx_{2n}, Tz))$,

letting $n \to \infty$, we get

$p(z, Bz) \leq \phi(p(z, Bz) < p(z, Bz)$ yields $z = Bz = Tz$.

Hence as in the proof of the theorem 3.1, z is the unique common fixed point of A, B, S and T. This completes the proof.

Remark. If we take $A = B$, $S = T$ = identity map and $\phi(\alpha) = \alpha k$, $k = 1$, $0 < \alpha < 1$ in our results then we get the result of Acharya [1].

Acknowledgement : The second author thanks UGC and Department of Higher Ed. of M.P. Govt. (India) for giving an opportunity to work as TRF.

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