Mathematical Forum
Vol. XIV, 2001-2002 pp 53-59

KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD FOR TIME-DEPENDENT NONLINEAR SYSTEMS WITH DAMPING

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ABSTRACT

Second-order time-dependent nonlinear differential system modeling damped oscillatory process is considered. The method is an extension of modified Krylov-Bogoliubov-Mitropolskii [1, 2, 7] method.

1. INTRODUCTION

Perturbation solution of a second-order time-dependent nonlinear system is found based on the modified Krylov-Bogoliubov-Mitropolskii (KBM) [1,2,7] method. The method was extended by Popov [3] to a damped nonlinear system (non-autonomous) and later some authors e.g., Mendelson [4], Bojadziev [5] and Murty [6] rediscovered the Popov's results. Shamsul [7] and Shamsul et al [8] further investigated nonlinear systems with strong damping effect and almost critical damping effect based on a critically damped solution [9]. In these cases Popov's [3] or Mendelson's [4] solution was unable to give desired results. Bojadziev [5] actually extended Popov's [3] solution in some biological and biochemical systems. Bojadziev [10] studied a damped forced nonlinear system. Arya and Bojadzeiv [11] also studied some time-dependent nonlinear systems. However, their solution (obtained in [11] gives desired results only for some significant damping forces much smaller than the critical damping force. The aim of this paper is to find an approximate solution of a time-dependent nonlinear system in which a strong linear damping force acts.

2. THE METHOD

Consider a weakly nonlinear system governed by the second-order differential equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}) + \varepsilon\Omega(t), \quad (1)$$

where over-dots denote differentiation with respect to t, ε is a small parameter, $k > 0$, $\omega > 0$, f is a nonlinear function and Ω is an external forcing term. Usually, Ω is periodic ; but Arya and Bojadziev's [10] considered a general case of the forcing term that Ω is bounded.

When $\varepsilon = 0$, (1) has two eigen-values, namely, $-k \pm i\omega_0$, $\omega_0^2 = \omega^2 - k^2$. Therefore, the unperturbed solution of (1) is

$$x(t,0) = a_0 e^{-kt} \cos(\omega_0 t + \varphi_0), \quad (2)$$

where a_0 and φ_0 are two arbitrary constants.

We seek a solution of (1) that reduces to (2) as the limit $\varepsilon \to 0$. We start with the solution

$$x(t,\varepsilon) = \exp(-kt + \alpha)\cos(\omega_0 t + \varphi) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \varepsilon^3 \ldots, \quad (3)$$

where $\alpha(t)$ and $\varphi(t)$, satisfy the first order differential equations

$$\dot{\alpha} = \varepsilon A_1(t) + \varepsilon^2 A_2(t) + \varepsilon^3 \ldots,$$
$$\dot{\varphi} = \varepsilon B_1(t) + \varepsilon^2 B_2(t) + \varepsilon^3 \ldots, \quad (4)$$

Confining only to the first few terms, 1, 2, ... m, in the series expansions of (3) and (4), we evaluate the functions $u_1, u_2, \ldots; A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ such that $\alpha(t)$ and $\varphi(t)$ appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a low order, usually the first [6]. In order to determine these functions, it was early imposed by Krylov, Bogoliubov and Mitropolskii [1,2] that the functions $u_1, u_2, \ldots$ do not contain first harmonic terms and later this assumption was strictly followed by Popov [3], Mendelson [4], Bojadziev [5] and Murty [6]. Shamsul [7] and Shamsul et al [8] proved that Krylov, Bogoliubov and Mitropolskii's [1,2] assumption is incorrect for the nonlinear system with large damping effects, especially when the damping force approaches toward the ciritical damping force.

Now differentiating (3) twice with respect to t, substituting for the derivatives $\dot{x}$, $\ddot{x}$ and x in (2), utilizing relations (4) and comparing the coefficients of ε, we obtain

$$\exp(-kt + \alpha)\,[(\dot{A}_1 - 2\omega_0 B_1)\cos(\omega_0 t + \varphi) - (2\omega_0 A_1 + \dot{B}_1)\sin(\omega_0 t + \varphi)] + \ddot{u}_1 + 2k\dot{u}_1 + \omega^2 u_1 = -f^{(0)} + \Omega, \quad (5)$$

where, $f^{(0)} = f(x_0, \dot{x}_0)$ and $x_0 = \exp(-kt + \alpha)\cos(\omega_0 t + \varphi)$.

In order to solve (5) for A_1, B_1 and u_1, we substitute $u_1 = v_1(t) + w(t)$ into (5) and separate it for A_1, B_1, v_1 and w as :

$$\exp(-kt + \alpha) [\dot{A}_1 - 2\omega_0 B_1) \cos(\omega_0 t + \varphi) - (2\omega_0 A_1 + \dot{B}_1) \sin(\omega_0 t + \varphi)] + \ddot{v}_1 + 2k\dot{v}_1 + \omega^2 v_1 = -f^{(0)}, \quad (6)$$

and

$$\ddot{w} + 2k\dot{\omega} + \omega^2 w = \Omega. \quad (7)$$

Equation (7) can be easily solved for w when Ω is specified. Followed by Popov's [3] assumption, Arya and Bojadziev [11] solved (6) under the restriction that v_1 excludes first harmonic terms. Shamsul [7] carefully investigated the nonlinear systems for different damping forces and observed that Krylov, Bogoliubov and Mitropolskii's [1,2] assumption is correct for certain damping effect, which is much smaller than the critical damping force. It is interesting to note that increasing with damping force, the error(s) of Popov's [3] or Mendelson's [4] solution is being increased and after a certain damping force, the solution gives incorrect results. In Shamsul's [7] and Shamsul et al [8] papers, perturbation solutions were found in which some first harmonic terms are involved in the so-called correction term, v_1. However, Shamsul's [7] and Shamsul et al's [8] previous solutions show good agreement with numerical solutions respectively for large and near to the critical damping forces. as a limiting approach, the solution obtained in [8] reduces to an exact critically damped solution found by Shamsul [9].

In accordance to Krylov, Bogoliubov and Mitropolskii's [1,2] assumption, $f^{(0)}$ be expanded in a Fourier series

$$f^{(0)} = \sum_{n=0}^{\infty} F_n \cos n(\omega_0 t + \varphi) + G_n \sin n(\omega_0 t + \varphi), \quad (8)$$

and then substituting the functional values of $f^{(0)}$ into (6) and comparing equal harmonic terms from both sides, we obtain

$$\exp(-kt + \alpha)(\dot{A}_1 - 2\omega_0 B_1) = -F_1, \quad (9)$$

$$\exp(-kt + \alpha)(2\omega_0 A_1 + \dot{B}_1) = G_1, \quad (10)$$

$$\ddot{v}_1 + 2k\dot{v}_1 + \omega^2 v_1 = - F_0 - \sum_{n=2}^{\infty} F_n \cos \text{II}\, n(\omega_0 t + \varphi) + G_n \sin \text{II}\, n(\omega_0 t + \varphi). \quad (11)$$

The particular solution of (9)-(11) gives three unknown functions A_1, B_1, and v_1 which complete (with w) the first order formal KBM solution of (1) found by Arya and Bojadziev [11].

On the contrary, in accordance to Shamsul's [7] assumption the corresponding equations of (9)-(11) become

$$\exp(-kt+\alpha)(\dot{A}_1 - 2\omega_0 B_1) = -F_1 \cos^2\varphi, \tag{12}$$

$$\exp(-kt+\alpha)(2\omega_0 A_1 + \dot{B}_1) = G_1 \cos^2\varphi, \tag{13}$$

and

$$\ddot{v}_1 + 2k\dot{v}_1 + \omega^2 v_1 = -F_0 - F_1 \cos(\omega_0 t + \varphi)\sin^2\varphi - G_1 \sin(\omega_0 t + \varphi) \sin^2\varphi \ldots, \tag{14}$$

The particular solutions of (12)–(14) again gives three unknown functions A_1, B_1 and v_1 ; but it is difficult to solve (12)-(14). In general, the right hand sides of (12)-(14) contain amplitudes and phase variables, which are also functions of t. However, we shall able to solve the equations (12)-(14) if the amplitude and phase variables, α and φ are assumed to be time-independent in the right hand sides of the equations (see [7,8,9,12]).

3. Example

As an example of the above procedure, we consider the Duffing's equation with an external forcing term and damping

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3 + \varepsilon E e^{-pt} \cos qt. \tag{15}$$

Here, $f^{(0)} = \frac{1}{4}\exp(-3kt + 3\alpha)\,[3\cos(\omega_0 t + \varphi) + \cos 3.(\varphi_0 t + \varphi)]$, or only nonzero coefficients are $F_1 = \frac{3}{4}\exp(-3kt + 3\alpha)$ and $F_3 = \frac{1}{4}\exp(-3kt + 3\alpha)$. Now substituting these values of F_1 and F_3 into (12)-(14), replacing α and φ in the right hand sides by their respective values obtained in the linear case and then solving them, we obtain

$$A_1 = \frac{3k\exp(-2kt+2\alpha_0)\cos^2\varphi_0}{8\omega^2}, \quad B_1 = \frac{3\omega_0 \exp(-2kt+2\alpha_0)\cos^2\varphi_0}{8\omega^2} \tag{16}$$

and

$$u_1 = -\frac{\exp(-3kt+3\alpha_0)}{16\omega^2} \times \left(\frac{(k\cos(\omega_0 t+\varphi_0) - \omega_0 \sin(\omega_0 t + \varphi_0))\sin^2\varphi_0}{k} + \frac{(k^2 - 2\omega_0^2)\cos 3(\omega_0 t + \varphi_0) - 3k\omega_0 \sin 3(\omega_0 t + \varphi_0)}{k^2 + 4\omega_0^2}\right). \tag{17}$$

Substituting the values of A_1 and B_1 form (16) into (4) and solving the equations, we obtain

$$\alpha = \alpha_0 + \frac{3\varepsilon \exp(2\alpha_0)\cos^2\varphi_0}{16\omega^2} \times (1-e^{-2kt}),$$

$$\varphi = \varphi_0 + \frac{3\varepsilon\omega_0 \exp(2\alpha_0)\cos^2\varphi_0}{16k\omega^2} \times (1-e^{-2kt}). \quad (18)$$

For the equation (15), the solution of equation (7) become

$$w = e^{-pt} \times \frac{(p^2 + 2kp + \omega^2 - q^2)\cos qt + 2q\,(k+p)\sin qt}{(p^2 + 2kp + \omega^2 - q^2)^2 + 4q^2\,(k+p)^2} \quad (19)$$

Therefore, the first approximate solution of (15) is

$$x(t,\varepsilon) = \exp(-kt + \alpha)\cos(\omega_0 t + \varphi) + \varepsilon\,(v_1 + w), \quad (20)$$

where α and φ are given by (18) and u_1 is given by (17).

4. Arya and Bojadziev's Solution

We can at once solve (9)-(11) for Arya and Bojadziev's [11] solution. Substituting the values of F_1 and F_3 into (9)-(11) and then solving them, we obtain

$$A_1 = \frac{3k \exp(-2kt + 2\alpha)}{8\omega^2},\ B_1 = \frac{3\omega_0 \exp(-2kt + 2\alpha)}{8\omega^2}, \quad (21)$$

and

$$u_1 = -\frac{\exp(-3kt + 3\alpha)}{16\omega^2} \times \frac{(k^2 - 2\omega_0^2)\cos 3(\omega_0 t + \varphi) - 3k\omega_0 \sin 3(\omega_0 t + \varphi)}{k^2 + 4\omega_0^2} \quad (22)$$

Substituting the values of A_1 and B_1 form (21) into (4) and integrating with respect to t, one obtains

$$\alpha = \frac{\alpha_0}{\sqrt{1 + \dfrac{3\varepsilon\alpha_0^2\,(e^{-2kt} - 1)}{8\omega^2}}},\ \varphi = \varphi_0 - \frac{\omega_0}{2k}\ln\left(1 + \frac{3\varepsilon\alpha_0^2\,(e^{-2kt} - 1)}{8\omega^2}\right). \quad (23)$$

Thus (20) represents also Arya and Bojadziev's [1] solution of (15), where α and φ are given by (23) and u_1 is given by (22).

5. Results and discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to a recent work [12]. First, solution (20) in where α, φ are computed by (18) and u_1 is computed by (17), is compared to the numerical solution (generated by Runge-Kutta fourth-order formula) with initial conditions $[x(0) = 1, \dot{x}(0) = 0]$ for $k = \frac{\sqrt{3}}{2}$, $\omega_0 = \frac{1}{2}$, $E = 1$, $p = 0$, $q = \sqrt{2}$ and $\varepsilon = 0.2$ in

Fig 1(a). Then Arya and Bojadziev's [11] solution has been compared to numerical solution in Fig. 1(b). By comparing the results of the figures, we can tell that the new solution (concerned of this paper) is better than the previous solution obtained by Arya and Bojadziev [11].

6. Conclusion

A simple formula is presented for obtaining the approximate solutions of a nonlinear system governed by a second-order nonlinear non-autonomous differential equation. The solution shows a good coincidence with numerical solution for certain damping effect while the formal KBM solution (early found by Arya and Bojadziev [11]) does not satisfy the numerical solution nicely. On the contrary, Arya and Bojadziev's [11] solution is useful for small damping one. In the case of significant damping force, both solutions (obtained in [11] and in this paper) give desired results.

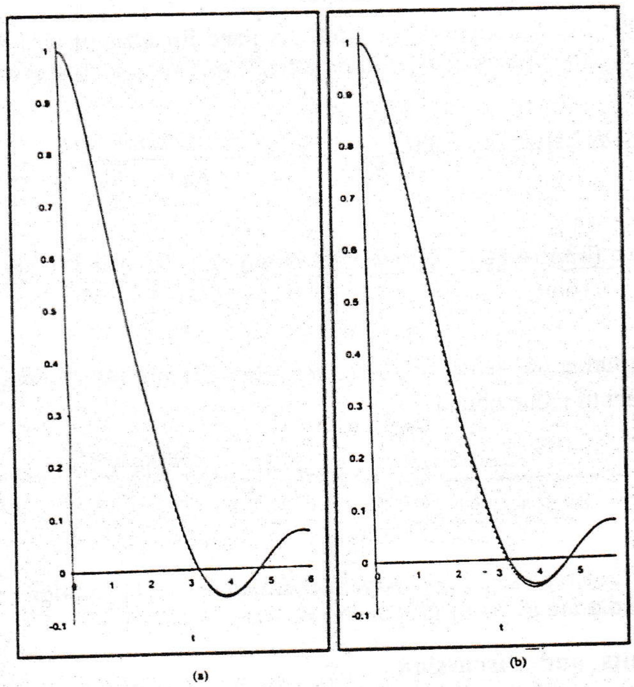

Fig. 1 (a) Perturbation solution (20) of Duffing equation (15) is presented by solid line in where α and φ are evaluated by (18) for $k = \frac{\sqrt{3}}{2}$, $\omega = \frac{1}{2}$, $E = 1$, $p = 0$, $q = \sqrt{2}$ and $\varepsilon = 0.2$ with initial conditions [$x(0) = 1$, $\dot{x}(0) = 0$]. (b) Arya and Bojadziev's [11] solution of (15) is presented by also solid line in where α and φ are evaluated by (23) for same values of k, ω_0, E, p, q, ε and with same initial conditions. Corresponding numerical solution is given by dashed line in both figures.

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