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AN EXISTENCE THEOREM FOR NONLINEAR MIXED INTEGRODIFFERENTIAL INCLUSIONS IN BANCH SPACES

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ABSTRACT

In this paper we prove the existence of mild solutions of nonlinear mixed integrodifferential inclusions in Banch spaces. The results are obtained by using the resolvent operators and a fixed point theorem for multivatued maps on locally convex topological spaces.

Keywords : Integrodifferential inclusion, convex multivalued map, resolvent operator, fixed point theorem.

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1. Introduction

The existence of mild, strong and classical solutions for differential and integrodifferential equations in abstract spaces have been studied by several authors [2,4-5, 12-14]. The existence of a resolvent operators for an integrodifferential equations in Banach spaces has been established by Grimmer [8]. Based on [8] Lin and Liu[10] studied the existence of mild solutions of the semilinear integrodifferential equations with nonlocal conditions by using the Banach fixed point theorem. Balachandran and Sakthivel [3] studied the existence theorem for nonlinear integrodifferential equations in Banach spaces. Avgerinos and Papageorgiou [1], Papageorgiou [15,16], and Benchohra [7] discussed the existence of solutions for first order differential inclusions on unbounded intervals. The purpose of this paper is to prove the existence of mild solutions for a nonlinear mixed integrodifferential inclusion of the form

$$\frac{du}{dt} - A\,[u(t) + \int_0^t F(t-s)\,u(s)\,ds] \in G\,(t,u, \int_0^t k(t,s,u(s))ds, \qquad (1)$$

$$\int_0^T h(t,s,u(s))ds),\ t \in I = [0,\infty],$$

$$u(0) = u_0,$$

where $G : I \times X \times X \times X \to 2^X$ is a bounded, closed, convex multivalued map $k: \Delta \times X \to X$, $h : \Delta \times X \to X$, are given functions, where $\Delta = \{(t,s) : 0 \leq s \leq t < \infty\}$, $u_0 \in X$, $F(t) : Y \to Y$, $AF(.)u(.) \in L^1(I,X)$, $F(t) \in B(X)$, $t \in I$ and for $u \in X$, $F'(t)\,u$ is continuous in $t \in I$, where $B(X)$ is the space of all bounded linear operators on X and Y is the Banach space formed from $D(A)$, the doman of A endowed with the graph norm. T is a real constant, A is the infinitesimal generator of a strongly continuous semigroup in a Banach space X. The method we are going to use is to reduce the problem (1) to search for fixed points of a suitable multivalued map on the Frechet space $C(I,X)$ and we make use of a fixed point theorem due to Ma [11] for multivalued maps in locally convex topological spaces.

2. Preliminaries

In this section we introduce the notations, definitions and preliminary facts from multivalued analysis which are used throughout the paper. Let m be a positive integer and $I_m = [0,m]$. $C(I,X)$ is the liner metric Frechet space of continuous functions from I into X with the metric

$$d(u,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\,\|u - z\|_m}{} \quad \text{for each } u,z \in C(I,X),$$

where $\|u\|_m = \sup\{\|u(t)\| : t\ I_m\}$. B(X) denotes the Banach space of bounded linear operators from X into X. A measurable function $u : I \to X$ is Bochner intergrable if and only if $|u|$ is Lebesgue intergrable. Let $L^1(I,X)$ denote the Banach space of continuous functions $u : I \to X$ which are Bochner intergrable normed by

$$\|u\|_{L^1} = \int_0^{\infty} \|u(t)\| : dt,$$

and U_p is a neighbourhood of 0 in $C(I,X)$ defined by

$$U_p = \{u \in C(I,X) : \|u\|_m \leq p\}$$

for each $m \in N$. The convergence in $C(I,X)$ is the uniform convergence on compact intervals, that is $u_j \to u$ in $C(I,X)$ if and only if for each m, $\|u_j - u\|_m \to 0$ in

$C(I_m, X)$ as $j \to \infty$. $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (that is, $\sup_{x \in B} \{\sup\{\|u\| : u \in G(x)\}\} < \infty$). G is called upper semi continuous on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open subset B of X containing $G(x_0)$, there exists an open neighourhood A of x_0 such that $G(A) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with non empty compact values, then G is upper semi continuous if and only if G has a closed graph (i.e.,$x_n \to x_0$, $u_n \to u_0$, $u_n \in Gx_n$ imply $u_0 \in Gx_0$. Now we shall define the solution of the problem (1).

Definition 2.1. A continuous solution $u(t)$ of the integral inclusion

$$u(t) \in R(t)u_0 + \int_0^t R(t-s)G\left(s,u,\int_0^s k(s,\tau,u(\tau))d\tau, \int_0^T h(s,\tau,u,(\tau))d\tau\right)ds$$

is called mild solution of (1) on I, where $R(t)$ is a resolvent operator of (1) with $G \equiv 0$ and $R(t) \in B(X)$ for $t \in I$ satisfying the following conditions (see[8]) :

(a) $R(0) = I$ (the identity operator on X),

(b) for all $x \in X$, $R(t)x$ is continuous for $t \in I$.

(c) $R(t) \in B(Y)$, $t \in I$. For $y \in Y$, $R(t)y \in C^1([0,b],X) \cap C([0,b],Y)$ and

$$\frac{d}{dt} R(t)y = A\left[R(t)y + \int_0^t F(t-s)\, R(s)\, y ds\right]$$

$$= R(t)\, Ay + \int_0^t R(t-s)\, AF(s)yds,\ t \in I.$$

We assume the following conditions :

(i) $G : I \times X \times X \times X \to BCC(X)$ is measurable with respect to t for each $u \in X$, upper semi continuous with respect to u for each $t \in I$ and for each $u \in C(I,X)$ the set $S_{G,u} = \{g \in L^1(I;R) : g(t) \in G(t,u(t), \int_0^t k(t,s,u(s))ds, \int_0^t h(t,s,u(s))ds)$ for a.e. $t \in I\}$ is non empty.

(ii) There exists functions $a(t)$, $b(t) \in C(I;X)$ such that

$|\int_0^t k(t,s,u)ds| \le a(t)\, \|u\|$ and $|\int_0^T h(t,s,u)ds| \le b(t)\, \|u\|$ for a.e $t,s \in I, u \in X$.

(iii) The resolvent operator $R(t)$ is compact such that $\max_{t>0} \|R(t)\| \le M$, where $M > 0$.

(iv) There exists a functions $\alpha(t) \in L^1(I;R_+)$ such that

$$\|G(t,u,v,w)\| \leq \alpha(t)\,\Omega(\|u\| + \|v\| + \|w\|)$$

for a,e $t \in I$, $u \in X$, where $\Omega : R_+ \to (0,\infty)$ is continuous, increasing function satisfying $\Omega(a(t)x + b(t)y) \leq a(t)\Omega(x) + b(t)\Omega(y)$ and

$$M\int_0^m \alpha(s)(1 + a(s) + b(s))ds < \int_c \frac{du}{\Omega(u)}$$

for each m where $c = M\|u_0\|$.

(v) For each neighbourhood U_p of 0, $u \in U_p$ and $t \in I$ the set

$$\{R(t)u_0 + \int_0^t R(t-s)\,g(s)\,ds,\ g \in S_{G,u}\}$$

is relatively compact.

Lemma 2.1 [9]. Let I be a compact real interval and X be a Babach space. Let G be a multivalued map satisfying (i) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator

$$\Gamma \circ S_G : C(I,X) \to X,\ (\Gamma \circ S_G)(y) = (S_{G,y})$$

is upper semi continuous in $C(I,X) \times C(I,X)$.

Lemma 2.2 [11]. Let X be a locally convex space. Let $N : X \to 2^X$ be a compact convex valued, upper semi continuous multivalued map such there exists a closed neighbourhood U_p of 0 for which $N(U_p)$ is a relatively compact set for each positive integer p. If the set $\zeta = \{y \in X : \lambda y \in N(y)\}$ for some $\lambda > 1$ is bounded, then N has a fixed point.

Remark [9]. If $\dim X < \infty$ and I is a compact real interval, then for each $u \in C(I, X)$, $S_{G,u}$ is nonempty.

3. Main Result

Theorem 3.1. If the assumptions (i)-(v) are satisfied, then the initial value problem (1) has at least one mild solution on I.

Proof. A solution to (1) is a fixed point for the multivalued map

$$N : C(I, X) \to 2^{C(I,X)}$$

defined by

$$N(u) = \{u \in C(I, X) : y(t) = R(t)u_0 + \int_0^t R(t-s)g(s)ds,\ g \in S_{G,u}\},$$

where

$$S_{G,u} = \{g \in L^1(I, X) : g(t) \in G(t,u,\int_0^t k(t,s,u(s))ds, \int_0^T h(t,s,u(s)ds) \text{ for a.e } t \in I\}.$$

First we shall prove $N(u)$ is convex for each $u \in C(I, X)$. Let $y_1, y_2 \in N(u)$, then there exists $g_1, g_2 \in S_{G,u}$ such that

$$y_i(t) = R(t)u_0 + \int_0^t R(t-s)g_i(s)ds, i = 1,2, t \in I$$

Let $0 \leq k_1 \leq 1$, then for each $t \in I$ we have

$(k_1y_1+(1-k_1)y_2)t = R(t)u_0 + \int_0^t R(t-s)(k_1g_1(s)+(1-k_1)g_2(s))ds.$

Since $S_{G,u}$ is convex, thus $ky_1+(1-k)y_2 \in N(u)$. Hence $N(u)$ is convex for each $u \in C(I, X)$.

Let $U_p = \{u \in C(I, X); \|u\| \leq p\}$ be a neighbourhood of 0 in $C(I, X)$ and $u \in U_p$, then for each $y \in N(u)$ there exists $g \in S_{G,u}$ such that for $t \in I$, we have

$$\begin{aligned}\|y(t)\| &\leq \|R(t)\|\,\|u_0\| + \int_0^t \|R(t-s)\|\,\|g(s)\|ds \\ &\leq M\|u_0\| + M\int_0^t \alpha(s)\,\Omega(\|u\| + a(t)\|u\| + b(t)\|u\|)ds \\ &\leq M\|u_0\| + M\int_0^t \alpha(s)(\Omega(\|u\|) + a(t)\,\Omega(\|u\|) + b(t)\Omega(\|u\|))ds \\ &\leq M\|u_0\| + M\int_0^t \alpha(s)(1+a(s)+b(s))\Omega(\|u\|)ds \\ &\leq M\|u_0\| + M\|\alpha\|_{L^1(I_m)}\,\|(1 + a(s) + b(s))\| \sup_{u\in U_p} \Omega(\|u\|)\end{aligned}$$

Hence $N(U_p)$ is bounded in $C(I,X)$ for each positive integer p.

Next we shall prove $N(U_p)$ is equicontinuous set in $C(I,X)$ for each positive integer p. Let $t_1, t_2, \in I_m$, $t_1 < t_2$ then for all $h \in N(u)$, $u \in U_p$, we have

$$\begin{aligned}\|y(t_1) - y(t_2)\| &\leq \|(R(t_2) - R(t_1))\, u_0\| + \left\|\int_0^{t_2} (R(t_2-s) - R(t_1-s))g(u)ds\right\| \\ &\quad + \left\|\int_{t_1}^{t_2} R(t_1-s)g(u)ds\right\| \\ &\leq \|(R(t_2) - R(t_1)\, u_0\| + \left\|\int_0^{t_2}(R(t_2 - s) - R(t_1 - s))g(u)ds\right\| \\ &\quad + M(t_2 - t_1)\int_0^{m} \|g(u)\|ds.\end{aligned}$$

Hence by Ascoli-Arzela theorem we conclude that $N(U_p)$ is relatively compact in$C(I,X)$.

Now we shall prove that N is upper semi continuous.

Let $u_n \to u_*$, $y_n \in N(u_n)$ and $y_n \to y_0$. We shall prove that $y_0 \in N(u_*)$, $y_n \in N(u_n)$ means that there exists $g_n \in S_{G,u_n}$ such that

$$y_n(t) = R(t)\, u_0 + \int_0^t R(t-s)\, g_n(s)ds,\ t \in I.$$

We must prove that there exists $g_0 \in S_{G,u}$ such that

$$y_0(t) = R(t)\, u_0 + \int_0^t R(t-s)\, g_0(s)ds,\ t \in J. \qquad (2)$$

The idea is then to use the fact that $y_n \to y_0$; and $y_n - R(t)u_0 \in \Gamma(S_{G,u})$ where

$$(\Gamma g)(t) = \int_0^t R(t - s)g(s)ds,\ t \in I.$$

So we consider the functions u_n, $y_n - R(t)\, u_0, g_n$ defined on the interval $[k,k+1]$ for any $k \in N \cup \{0\}$. Then using Lemma 2.1, in this case we are able to say that (2) is true on the compact interval $[k,k+1]$, that is,

$$[\, y_0(t)]_{[k,k+1]} = R(t)u_0 + \int_0^t R(t - s)\, g_0^k\,(s)ds$$

for a suitable L^1- selection g_0^k of $G(t,u,\int_0^t k(t,s,u)ds, \int_0^T h(t,s,u)ds)$ on the interval $[k,k+1]$. Let $g_0(t) = g_0^k\,(t)$ for $t \in [k,k+1]$. We obtain that g_0 is an L^1- selection and

(2) will satisfied. Clearly we have $\|(y_n - R(t)u_0) - (y_0 - R(t)u_0)\|\infty \to 0$ as $n \to \infty$. Consider for all $k \in N \cup \{0\}$, the mapping

$$S_G^k : C([k,k+1], X) \to L^1([k,k+1], X),$$

$$y \to S_{G,y}^k = \{g \in L^1([k,k+1], X) : g(t) \in G(t,u \int_0^t k(t,s,u)ds, \int_0^T h(t,s,u)ds)$$

for a.e. $t \in ([k,k+1]\}$.

Now we consider the linear continuous operators

$$\Gamma_k : L^1([k,k+1], X) \to C([k,k+1], X),$$

$$g \to \Gamma_k(g)(t) = \int_0^t R(t-s)g(s)ds.$$

From Lemma 2.1 it follows that $\Gamma_k o S_G^k$ is upper continuous for all $k \in N \cup \{0\}$. Moreover, we have

$$(y_n(t) - R(t)u_0)|_{[k,k+1]} \in \Gamma_k(g_{G,u_n}^k)$$

and $u_n \to u_*$. From Lemma 2.1 we have $(y_0(t) - R(t)u_0)|_{[k,k+1]} \in \Gamma_k(g_{G,u*}^k)$,

$$(y_0(t) - R(t)u_0)|_{[k,k+1]} = \int_0^t R(t-s)\, g_0^k(s)\, ds \text{ for some } g_0^k \in g_{G,u*}^k.$$

Hence the functions g_0 defined on I by $g_0(t) = g_0^k(t)$ for $t \in [k,k+1]$ is in $S_{G,u*}$. Therefore $N(U_p)$ is relatively compact for each p and N is upper semi continuous with convex closed values. Finally we prove the set $\zeta = \{u \in C(I,X);\ \lambda u \in Nu\}$ for some $\lambda > 1$ is bounded.

Let $\lambda u = Nu$ for some $\lambda > 1$ then there exists $g \in S_{G,u}$ such that

$$u(t) = \lambda^{-1}R(t)u_0 + \lambda^{-1} \int_0^t R(t-s)g(s)ds,\ t \in I,$$

$$\|u(t)\| \le M\|u_0\| + M\int_0^t \alpha(s)(1+a(s)+b(s))\Omega(\|u\|)ds.$$

Let $\upsilon(t) = M\|u_0\| + M\int_0^t \alpha(s)(1+a(s)+b(s))\Omega(\|u\|)ds$,

then we have $\upsilon(0) = M\|u_0\| = c$ and $\|u(t)\| \le \upsilon(t), t \in I_m$. Using the increasing character of Ω we get

$$\upsilon'(t) \le M\alpha(t)(1+a(t)+b(t))\,\Omega(\upsilon(t)),\ t \in I_m.$$

This proves for each $t \in I_m$ that

$$\int_{\upsilon(0)}^{\upsilon(t)} \frac{du}{\Omega(u)} \le M\int_c^m \alpha(s)(1+a(t)+b(t))ds < \int_c^\infty \frac{du}{\Omega(u)}.$$

This inequality implies that there exists a constant M_0 such that $\upsilon(t) \le M_0, t \in I_m$, and hence $\|u\|_\infty \le M_0$ where M_0 depends on m and on the functions α, a, Ω. Hence ζ is bounded. Thus by Lemma 2.2 N has a fixed point which is a mild solution of (1).

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