Mathematical Forum
Vol.XV, 2002-2003

THE COMPLEMENTARY FUZZY TOPOLOGY

V. SEENIVASAN
Department of Mathematics,
Jawahar Science College,
Neyveli - 607 803,
Tamilnadu, India.

AND

G. BALASUBRAMANIAN
Ramanujan Institute for advance
Study in Mathematics
University of Madras
Chepauk, Madras - 600005

ABSTRACT

We introduce the concept of the Complementary Fuzzy Topology by making use of the interior and the complement operator. We discuss and study several interesting properties of this newly introduced fuzzy topology.

Keywords : Complementary fuzzy topology, Dense fuzzy set, Nowhere dense fuzzy set, Fuzzy connected, Fuzzy Hausdorff.

1. Introduction

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy set by Zadeh [7]. Fuzzy sets have applications in many fields such as informations [4] and control [5]. The theory of fuzzy topological space was introduced and developed by Chang [2] and since then various notions in classical topology have been extended to fuzzy topological space. The concept of complementary toplogy was introduce in Norman Levine [3]. In this paper we introduce the concept of complementary fuzzy topology by making use of the interior and the complementary operator and we discuss and study several interesting properties of this newly introduced fuzzy topology.

2. PRELIMINARIES

By a fuzzy topological space we shall mean a non-emply set X together with a fuzzy topology T (in the sense of Chang [2]) and denote it by (X,T).
Let, $x \in X$, $t \in [0,1]$. A fuzzy point is a fuzzy set x_t defined by

$$x_t(y) = \begin{cases} 0 \text{ if } y \neq x \\ t \text{ if } y = x \end{cases}$$

The fuzzy point x_t is called the crisp-point. Let λ be fuzzy set in X and x_t a fuzzy point in X. We say that $x_t \in \lambda$ if and only if $x_t \leq \lambda$. Let (X,T) and (Y,S) be any two fuzzy topological spaces. Let f : (X,T) → (Y,S) be any function. For any $\lambda \in S$, we define $f^{-1}(\lambda)$ as follows $f^{-1}(\lambda)(x) = \lambda f(x)$ [6] for all $x \in X$. Also for any $\mu \in T$, we define $f(\mu)$ as follows.

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} f(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

A function f from a fuzzy topological space (X,T) to fuzzy topological space (Y,S) is said to be fuzzy continuous if for each fuzzy open set λ in S the inverse image $f^{-1}(\lambda)$ is fuzzy open set in (X,T). f is called fuzzy open if the image of each fuzzy open λ in (X,T) is fuzzy open set in (Y,S). Let (X,T) be any fuzzy topological space.Let λ be any fuzzy set in X. We define cl $\lambda = \wedge \{\mu/\lambda \leq \mu, 1 - \mu \in T\}$ and int $\lambda = \vee \{\mu/\mu \leq \lambda, \mu \in T\}$. For any fuzzy set λ in a fuzzy topological space, it is easy to see that 1-clλ = int $(1-\lambda)$[1]. Let (X,T) be any fuzzy topological space and let A be any non-empty subset of X. Let T/A = $\{\lambda/A : A \to I / \lambda \in T\}$. It is well known that T/A is a fuzzy topology on A and the pair (A,T/A) is called a fuzzy subspace of X.

3. The Complementary Fuzzy Topology

Let (X,T) be any fuzzy topological space and let $\mathcal{B}^* = \{\text{int } \lambda \mid 1-\lambda \in T\}$. It is clear that $\mathcal{B}^*$ is a base for a fuzzy toplogy T* on X and $T^* \subseteq T$. We shall call T* the Complementary fuzzy topology. cl and cl* will denote the closure operators and int and int* will denote the interior operators relative to T and T* respectively. The following proposition is easy to establish.

PROPOSITON 3.1 : Let (X,T) be a fuzzy topolgical space. Then $(T^*)^* = T^*$

Corollary 3.1 : Let (X,T) be any fuzzy topological space. Then there exists a fuzzy topology T_0 for X for which $T = T_0^*$ if and only if $T = T^*$.

PROPOSITON 3.2 : Let (X,T) be any fuzzy topological space and let T* be the complementary fuzzy topology for X. If $\lambda \in I^X$, then int cl $\lambda \leq$ int* cl* λ.

Proof : It is easy to see int cl $\lambda \leq$ cl $\lambda \leq$ cl* λ ......(1). But int cl $\lambda \in \mathcal{B}^* \subset T^*$ ......(2). Therefore int* (int cl λ) $\leq$ int* cl* λ [by (1)] that is int cl $\lambda \leq$ int* cl* λ [by (2)].

DEFINITION 3.1 : Let (X,T) be any fuzzy topological space and let λ be any fuzzy set in X. Then λ is called a dense fuzzy set if cl $\lambda = 1$ and λ is called a nowhere dense fuzzy set if int cl $\lambda = 0$.

PROPOSITON 3.3 : Let (X,T) be any fuzzy topological space. Then T* is indiscrete if and only if every fuzzy set λ in X is either T-dense or nowhere T-dense in X.

Proof : Now T* is indiscrete $\Leftrightarrow$ T* = {0,1} $\Leftrightarrow \mathcal{B}^*$ = {0,1) $\Leftrightarrow$ For every T-closed set λ, int $\lambda = 0$ or int $\lambda=1 \Leftrightarrow$ for any fuzzy set λ in X, we have int cl λ= 0 or int cl λ =1 $\Leftrightarrow$ every fuzzy set λ in X is such that λ is nowhere T-dense or T-dense.

Remarks : It is of interest to note that if T_1 and T_2 are fuzzy topologies for X such that $T_1 \subset T_2$, then in general $T_1^* \not\subset T_2^*$. The following example serves the purpose.

Example 3.1 : Let X = {a,b,c,} and T_1 = {0,f,g,h,1} and T_2 = {0,f,g,h,k,1} where f : X → [0,1] is such that f (a) = 1, f(b) = 0, f(c) =0 ; g : X → [0,1] is such that g(a) = g(b) = 0, g(c) = 1; h : X → [0,1] is such that h(a) = h(c) =1, h(b) = 0; and k : X → [0,1] is such that k(a) = k (b) = 1, k(c) = 0. It is easy to check that $T_1 \subset T_2$ but $T_1^* \not\subset T_2^*$.

Corollary 3.2 : Let (X,T) be a fuzzy topological space and suppose λ be any fuzzy set in X. Then (1) if λ is T^* - nowhere dense, then λ is T-nowhere dense (2) if X is T^*-first category. then X is of T-first category.

Proof follows from proposition 3.2.

PROPOSITION 3.4 : (X,T) is fuzzy connected if and only if (X,T^*) is fuzzy connected.

Proof : Suppose (X,T) is fuzzy connected. Since $T^* \subset T$, it follows that (X,T^*) is fuzzy connected. Conversely suppose (X,T^*) is not fuzzy connected. If (X,T^*) is not fuzzy connected, then there exists a fuzzy set λ such that it is fuzzy T-open and T-closed. It is easy to see that λ is both T^*-open and T^*-

closed which is a contradiction to the assumption. Therefore (X,T) is fuzzy connected.

PROPOSITION 3.5 : Let (X,T) and (Y,S) be fuzzy topological spaces and suppose that $f : X \to Y$ is continuous and open. Then for each fuzzy set λ in Y, we have

(1) $f^{-1}(\text{int}\lambda) = \text{int } f^{-1}(\lambda)$

(2) $f^{-1}(\text{cl}\lambda) = \text{cl } f^{-1}(\lambda)$

(3) $f^{-1}(\text{intcl}\lambda) = \text{intcl}f^{-1}(\lambda)$

Proof of (1) : Now int $\lambda \leq \lambda$ also $f^{-1}(\text{int } \lambda) \leq f^{-1}(\lambda)$ implies that $f^{-1}(\text{int } \lambda) \leq \text{int } f^{-1}(\lambda)$ ......(1). Similarly we can show that int $f^{-1}(\lambda) \leq f^{-1}(\text{int } \lambda)$ ......(2). Since int $f^{-1}(\lambda) = f^{-1}(\text{int } \lambda)$. Therefore (1) is verified.

Proof of (2) : $f^{-1}(\text{cl } \lambda) = f^{-1}[1\text{-int}(1\text{-}\lambda)] = 1\text{-} f^{-1}[\text{int}(1\text{-}\lambda)] = 1\text{- int }[1\text{-}f^{-1}(\lambda)] = \text{cl } f^{-1}(\lambda)$. Hence (2) is verified

Proof of (3) : Follows from (1) and (2).

PROPOSITION 3.6 : Let $f : (X,T) \to (Y,S)$ be fuzzy continuous and open map. Then $f : (X,T^*) \to (Y,S^*)$ is fuzzy continuous.

Proof : Let λ be any S-closed fuzzy set in Y. Then $f^{-1}(\text{int } \lambda) = \text{int } f^{-1}(\lambda)$ by proposition 3.5 and so $f^{-1}(\text{int}\lambda) \in T^*$.

Example 3.2 : If $f : X \to Y$ is fuzzy continuous and open relative to T and S, in general, f will not be open relative to T^* and S^* as shown by.

Let X = {a,b,} and let T = {0,1,f,g} where f(a) = 1, f(b) = 0; g(a) = 0, g(b) = 1. Clearly $T = T^*$. Suppose Y = {a,b} and S = {0,f,1}, then S^* = {0,1}. If f(x) = a for all $x \in X$, then f is fuzzy continuous and fuzzy open relative to T and S. But f is not open relative to T^* and S^*.

PROPOSITION 3.7 : Let $(X_\alpha, T_\alpha)_{\alpha \in \Delta}$ be a non-empty family of non-empty fuzzy topological spaces and let (X,T) denote the fuzzy product space. If $(X,T^{\neq})$ denote the fuzzy product of the family $(X_\alpha, T^*_\alpha)_{\alpha \in \Delta}$, then $T^{\neq} = T^*$.

Proof : We shall first show that $T^{\neq} \subseteq T^*$. Now for each $\alpha \in \Delta$, the projection map $P_\alpha : X \to X_\alpha$ is fuzzy continuous and fuzzy open relative to T and $T\alpha$ and so by proposition 3.6, P_α is fuzzy continuous relative to T^* and T^*_α. Since $T^{\neq}$ is the smallest fuzzy topology for X such that $P_\alpha : X \to X_\alpha$ is fuzzy continuous relative to T^*_α for all $\alpha \in \Delta$, it follows that $T^{\neq} \subseteq T^*$......(1). Conversely, let $\lambda^* \in T^*$. Then $\lambda^* = \text{int } \mu$, μ is T-closed. Therefore we have, $\lambda^* = \text{int } \mu = \vee \{ \wedge P_{\alpha j}^{-1}[\text{int cl } \lambda_{\alpha j}]\} = \vee$

int cl { $\wedge P_{\alpha j}^{-1}[\lambda_{\alpha j}]$}. But int cl $\lambda_{\alpha j} \in T^*_{\alpha j}$ and so $P_{\alpha j}^{-1}$[int cl $\lambda_{\alpha j}$] $\in T^{\neq}$. This implies that $\lambda^* \in T^{\#}$ .Therefore $T^* \subseteq T^{\neq}$. Hence $T^* = T^{\neq}$ the proposition.

Proposition 3.8 : Let (X,T) be any fuzzy topological space and let T^* be the complementary fuzzy topology. Let (Y,T/Y) be any fuzzy subspace of X. Then $(T/Y)^* = T^*/Y$.

Proof : A member of T/Y may be given as $\mu \wedge \chi_Y$ where $\mu \in T$. Now int y[1_Y - ($\mu \wedge \chi_Y$)] = 1_Y - Cl_Y. ($\mu \wedge \chi_Y$) = 1_Y - ($\chi_Y \wedge Cl_X \mu$) = $\chi_Y \wedge [1_X - (\chi_Y \wedge Cl_X \mu)] = \chi_Y \wedge [1_X - Cl_X \mu]$ = $\chi_Y \wedge int_X[1_X - \mu]$ where $\mu \in T$.

Definition 3.2 : Let (X,T) be any fuzzy topological space. (X,T) is called fuzzy Hausdorff if given any two district fuzzy points p and q in X there exists fuzzy open set $\lambda, \mu \in T$ such that $p \in \lambda$, $q \in \mu$ and $\lambda \leq 1-\mu$.

Proposition 3.9 : Let (X,T) be any fuzzy topological space and T^* be the complementary fuzzy topology. Then (X,T) is fuzzy Hausdorff if and only if (X,T*) is fuzzy Hausdorff.

Proof : If (X,T^*) is fuzzy Hausdorff then it follows easily that (X,T) is also fuzzy Hausdorff since$T^* \subseteq T$. To prove the converse, suppose (X,T) is fuzzy Hausdroff. Let p and q be two distinct fuzzy points in X. Then there exists fuzzy open sets λ, $\mu \in T$ such that $p \in \lambda, q \in \mu$ and $\lambda \leq 1-\mu$ ......(1). Hence it follows that $p \notin cl\mu$ and $q \notin cl\lambda$. Now put λ^*= int $(1-\mu) \wedge$ int (cl λ) and μ^* = int $(1-\lambda) \wedge$ int (cl μ).Clearly $\lambda^*, \mu^* \in T^*$ and $1-\mu^*$ = 1- [int $(1-\lambda) \wedge$ int cl μ] = [1-int$(1-\lambda)$] $\vee$ [1-int clμ] = cl $\lambda \vee$ cl $(1-\mu)$ = cl $(1-\mu) \geq$ int $(1-\mu) \geq \lambda^*$[since $\lambda \leq 1-\mu$] by (1). That is $1-\mu^* \geq \lambda^*$. It is easy to see that $p \in \lambda^*$, $q \in \mu^*$. This shows that (X,T^*) is fuzzy Hausdorff. Hence the proposition.

Corollary 3.3 : A fuzzy topological space (X,T) is fuzzy T_2 if and only if for any two distinct fuzzy points p and q of X, there exists fuzzy sets λ, $\mu \in T$ such that

(1) $1 - p \notin$ cl λ, $1 - q \notin$ cl μ

(2) cl λ + cl $\mu \leq 1$.

Proof : Let p and q be any two distinct fuzzy points and λ, $\mu \in T$ be such that $1 - p \notin$ cl λ, $1 - q \notin$ cl μ and cl λ + cl $\mu \leq 1$. Now $1 - p \notin$ cl λ implies $1 - p >$ cl λ. That is $p < 1 -$ clλ implies that $p \in 1 -$ clλ. Similarly one can see that $q \in$ 1-cl μ. Now clearly 1 - clλ and 1 - clμ are in T^* and from the assumption through clλ + cl $\mu \leq 1$, it follows that 1 - cl $\lambda \leq$ cl μ or 1 - cl $\mu \leq$ cl λ.This proves that (X,T^*) is fuzzy Hausdorff. Therefore by proposition 3.9, (X,T) is fuzzy Hausdorff. Conversely, suppose that (X,T) is fuzzy Hausdorff and p and q are any two distinct fuzzy points of X. Then by proposition 3.9, (X,T^*) is fuzzy Hasusdorff. Therefore there exists fuzzy set λ and μ which are T-fuzzy closed and for which $p \in \text{int}\,\lambda$, $q \in \text{int}\,\mu$ and $1-\text{int}\,\lambda \leq \text{int}\,\mu$. Then $1-p \notin cl\,(1-\lambda)$ and $1-q \notin cl\,(1-\mu)$. For $p \in \text{int}\,\lambda$ implies $p \leq \text{int}\,\lambda$ that is $-p \geq -\text{int}\,\lambda$ that is $1-p \geq 1-\text{int}\,\lambda = cl(1-\lambda)$ implies that $1-p \notin cl(1-\lambda)$. Similarly $1-q \notin cl(1-\mu)$. Now $cl(1-\lambda) + cl\,(1-\mu) = 1-\text{int}\,\lambda + 1-\text{int}\,\mu \leq \text{int}\,\mu + 1-\text{int}\,\mu = 1$. This proves the converse. Hence the corollary.

PROPOSITION 3.10 : Let T^* be the complementary fuzzy topology in (X,T). Then $T = T^*$ if and only if for each fuzzy point p such that $1-p \notin \lambda$, where λ is T-fuzzy closed, there exists fuzzy sets u_1 and u_2 in T such that $p \in u_1$, $\lambda \leq 1-u_2$ and $u_1 \leq 1-u_2$.

Proof : Assume $T = T^*$. Suppose p is a fuzzy point such that $1-p \notin \lambda$ where λ is T-fuzzy closed. Then $p < 1-\lambda \in T = T^*$. Hence there exists a T-fuzzy closed set u such that $p \in \text{int}\,u \leq 1-\lambda$. Put $u_1 = \text{int}\,u$ and $u_2 = 1-u$. Then $p \in u_1$, $\lambda \leq 1-\text{int}\,u$ and $1-u_2 = 1-(1-u) = u \geq \text{int}\,u = u_1$. Therefore $1-u_2 \geq u_1$. Sufficiency, we will show that $T \subseteq T^*$. Let $p \in \mu \in T$ and let $\lambda \in cl(\mu)$. Then there exists fuzzy sets u_1 and u_2 in T such that $p \in u_1, \lambda \leq 1-u_2$ and $u_1 \leq 1-u_2$. But $p \in 1-cl\,(u_2) = \text{int}\,(1-u_2) \leq \text{int}\,\lambda \leq \text{int}\,cl(\mu)$. since $1-cl\,(u_2) = (1-u_2) \in \mathcal{B}^*$. Therefore $p \in \text{int}\,(cl\,\mu) \in \mathcal{B}^* \subset T^*$, it follows that $p \in T^*$. Therefore $T \subseteq T^*$. Then $T = T^*$.

Corollary 3.4 : If (X,T) is fuzzy regular, then $T^* = T$.

The converse of corollary 3.4 is false as shown by

Example 3.3 : Let X = { a, b, c,} and T = {0, f, g, k, 1}. where $f : X \to [0,1]$ is such that f(a) = 1, f(b) = 0, f(c) = 0; $g : X \to [0,1]$ is such that g(a) = 0, g(b) = 1, g(c) = 0; $k : X \to [0,1]$ is such that k(a) = 1, k(b) = 1, k(c) = 0; $j : X \to [0,1]$ is such that j(a) = 0, j(b) = 1, j(c) = 1. Now (X,T) is not fuzzy regular since f and j cannot be separated by fuzzy open sets. It is easy to check that $T = T^*$ i.e., $T^* = \{1,g,f,k,0\}$.

PROPOSITION 3.11 : Let (X,T) be fuzzy topological space, $\lambda \in T$ and $Y \subset X$ is such that is χ_Y is T-dense in X. Then $cl(\lambda \wedge \mu) = cl\lambda$.

Proof : $cl\,(\lambda \wedge \mu) = cl\lambda \wedge cl\mu = cl\lambda \wedge 1 = cl\lambda$.

PROPOSITION 3.12 : Let (X,T) be a fuzzy topological space and let T^* be the complementary fuzzy topology. If $Y \subset X$ is such that χ_Y is T-dense in X, then $\mathcal{U}^* = T^*/Y$ where $\mathcal{U} = T/Y$.

Remark : In Proposition 3.12, the condition if $Y \subset X$ such that χ_Y is T-dense in X cannot be ommited as shown by

Example 3.4 : Let X = {a,b,c,} and let T = {0,f,g,k,1}, where $f : X \to [0,1]$ such that f(a)=1, f(b)=0, f(c)=0 ;$g:X \to [0,1]$ is such that g(a) = 0, g(b) = 1, g(c) =0; $k : X \to [0,1]$ is such that k(a) =1, k(b) = 1, k(c) = 0. Then $T^* = \{1,g,f,0\}$. Consider Y = {b,c} $\subset$ X then clearly χ_Y is not T-dense in X. But $(T/Y)^* \neq T^*/Y$.

PROPOSITION 3.13 : If (X,T) is fuzzy Hausdorff and fuzzy compact, then $T=T^*$.

Proof : A fuzzy Hausdorff compact space is fuzzy regular. By corollary 3.4, we get $T=T^*$.

Note : If (X,T) is fuzzy T_1 space, then (X,T^*) need not be fuzzy T_0 as shown by

Example 3.5 : Let X be an infinite set and suppose that T is fuzzy confinite topology , then $T^* = \{0,1\}$.

References :

[1] Azad, K.K. ,On Fuzzy Semicontinuity, Fuzzy Almost Continuity and Fuzzy weakly continuity, *J. Math. Anal. Appl.,* 82 (1981), 14-32.

[2] Chang. C.L., Fuzzy Topological Spaces, *J. Math. Anal.,* 24 (1968), 182-190.

[3] Norman Levine, The Complementary topology, *Rend. Circ. Mat. Palermo.,* 2 (1967), 247-251.

[4] Semets. P., The degree of belief in a fuzzy event, *Information Sciences,* 25 (1981), 1-19.

[5] Sugeno. M., An introductory survey of fuzzy control, *Information Sciences,* 36 (1985), 59-83.

[6] Warren. R.H., Neighbourhoods, base and continuity in fuzzy topological spaces, *Rocky Mountain J. Math.,* 8 (1978), 459-470.

[7] Zadeh. L.A., Fuzzy Sets, *Information and Control,* 8 (1965), 338-353.

***