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THREE DIMENSIONAL BOUNDARY LAYER EQUATIONS OVER A HORIZONTAL SURFACE

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Abstract :

This paper treats the three dimensional boundary layer equations over horizontal curvilinear surfaces. Here the similarity solution of the equations considered will be studied under some restricted conditions with some external velocities.

Key Words : Three dimension, Boundary layer, Similarity analysis.

Introduction : The general boundary layer equations for steady three dimensional laminar flow configuration deduced by Mager (1964) in absence of the body forces potential for an irrational motion, are considered as the basic equation in study. A three dimensional horizontal surface fixed in viscouse fluid horizontally is placed in a steady free stream. The surface is rigid and is not rotated. The general equations of boundary layer flow along that developable curved surface are setup in an approapriate system of orthogonal co-ordinates. Howarth (1951) obtained equations for the boundary layer flow over a general curved surface. He used a triply-orthpgonal co-ordinate system (ξ,η,ζ), such that ξ = constant and η = constant represent developable surfaces formed by the normal to the lines of curvature of the surface, and ζ = constant represents a surface parallel to the surface.

In fact the restriction to lines of curvature is unnecessary, as Howarth's equations are valid for any orthogonal system of curves ξ = constant, η = constant on the surface, as shown by Squire (1957).

Let us choose such an arbitrary system (ξ,η,ζ) and suppose that the lengt elements are $h_1 d\xi$, $h_2 d\eta$, $h_3 d\zeta$ in the ξ,η,ζ directions. Here h_3 is by definition function of ζ along, so that we may set $h_3 = 1$ and use ζ to measure distanc normal to the surface.

Governing Equations : If $u, v, w,$ are the velocity components in the ξ,η, directions repectively. The divergence and curl of $V=V(u,v,w)$ in the co-ordinate of ξ,η and ζ is

$$div\overline{V} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + \frac{\partial}{\partial \varsigma}(h_1 h_2 w)\right] \tag{1}$$

and

$$curl\overline{V} = \frac{1}{h_2}\left[\frac{\partial w}{\partial \eta} - \frac{\partial}{\partial \varsigma}(h_2 v)\right], \frac{1}{h_1}\left[\frac{\partial}{\partial \varsigma}(h_1 u) - \frac{\partial w}{\partial \xi}\right], \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}(h_2 v) - \frac{\partial}{\partial \eta}(h_1 u)\right] \tag{2}$$

The components of acceleration are

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \varsigma} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{wu}{h_1}\frac{\partial h_1}{\partial \varsigma}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \varsigma} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{wv}{h_2}\frac{\partial h_2}{\partial \varsigma} \tag{3}$$

$$\frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{v}{h_2}\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial \varsigma} - \frac{u^2}{h_1}\frac{\partial h_1}{\partial \varsigma} - \frac{v^2}{h_2}\frac{\partial h_2}{\partial \varsigma}$$

On making the usual boundary layer assumtions, with the additiona assumption that h_1 h_2 and all their first derivatives are 0(1), we have the followin equation of motion.

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \varsigma} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \upsilon\frac{\partial^2 u}{\partial \varsigma^2} \tag{4}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \varsigma}+\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}=\frac{1}{\rho h_2}\frac{\partial p}{\partial \eta}+\upsilon\frac{\partial^2 v}{\partial \varsigma^2} \quad (5)$$

$$\frac{\partial p}{\partial \varsigma}=0(1) \quad (6)$$

and the equation of continuity is

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}(h_2 u)+\frac{\partial}{\partial \eta}(h_1 v)+\frac{\partial}{\partial \varsigma}(h_1 h_2 w)\right]=0 \quad (7)$$

The above equations are in simplified form due to Mager (1964) in absences of angular velocity and body force potential. The length elements h_1, h_2, are not completely arbitrary.

We suppose that the potential flow in the mainstream outside the boundary layer is irrotational, so that if we denote by external velocities U_e, V_e in the directions ξ, η increasing, then the pressure gredient components be

$$\frac{U_e}{h_1}\frac{\partial U_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial U_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{V_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}=-\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} \quad (8)$$

$$\frac{U_e}{h_1}\frac{\partial V_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial V_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{U_e^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}=-\frac{1}{\rho h_2}\frac{\partial p}{\partial \eta} \quad (9)$$

The boundary condition on the orthogonal surface with no suction or injection are $u\to 0$, $v\to 0$. Similarity at the outer edge of the boundary layer (i.e.,$\varsigma\to\infty$), the velocity components taken on the external flow are $u\to U_e(\xi,\eta)$ and $v\to V_e(\xi,\eta)$.

The euqation (7) becomes

$$\frac{\partial}{\partial \xi}(h_2 u)+\frac{\partial}{\partial \eta}(h_1 v)+\frac{\partial}{\partial \varsigma}(h_1 h_2 w)=0 \quad (10)$$

From (4), (5), (8) and (9) we can write u - momentum and v - momentum equations for incompressible fluid are

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \varsigma}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial \eta}-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial \xi}-\upsilon\frac{\partial^2 u}{\partial \varsigma^2}=$$

$$\frac{U_e}{h_1}\frac{\partial U_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial U_e}{\partial \eta}+\frac{U_eV_e}{h_1h_2}\frac{\partial h_1}{\partial \eta}-\frac{V_e^2}{h_1h_2}\frac{\partial h_2}{\partial \xi} \tag{11}$$

and

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \varsigma}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial \eta}-\upsilon\frac{\partial^2 v}{\partial \varsigma^2}=$$

$$\frac{U_e}{h_1}\frac{\partial V_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial V_e}{\partial \eta}+\frac{U_eV_e}{h_1h_2}\frac{\partial h_2}{\partial \xi}-\frac{U_e^2}{h_1h_2}\frac{\partial h_1}{\partial \eta} \tag{12}$$

where υ is the kinematic viscosity of the fluid.

Similar solution : The boundary layer similarity solution for the three dimensiona case was studied by Hansen (1958). Here, we chance the variable (ξ,η,ς) to new set of variables (X,Y,ϕ), the relation between them be

$$X=\xi,\quad Y=\eta,\quad \phi=\frac{\varsigma}{\gamma(X,Y)} \tag{13}$$

Here γ(X,Y) can be throught of being proportional to the local boundar layer thickness. Let the two scalar functions $\Psi(\xi,\eta,\varsigma)$ and $\Phi(\xi,\eta,\varsigma)$ are define as the mass flow components within the boundary layer and in case c incompressible, flow, we may write

$$\Psi_\varsigma = h_2 u \tag{14}$$

$$\Phi_\varsigma = h_2 v \tag{15}$$

$$-(\Psi_\xi + \Phi_\eta) = h_1 h_2 w \tag{16}$$

Which satisfy the continuity equation (10). Guided by the idea of similarity procedure, we may put

$$\int_0^{\phi} \frac{u}{U(X,Y)} d\phi = f(X,Y,\phi) \tag{17}$$

and

$$\int_0^{\phi} \frac{v}{V(X,Y)} d\phi = g(X,Y,\phi) \tag{18}$$

In attempting separation of variables of $f(X,Y,\phi)$ and $g(X,Y,\phi)$, we assume

$$f(X,Y,\phi) = L(X,Y)\overline{f}(\phi)$$

and

$$g(X,Y,\phi) = M(X,Y)\overline{g}(\phi)$$

where $\overline{f}$ and $\overline{g}$ are functions of the single variable ϕ. From (13), (14) and (15) we obtain

$$\frac{1}{\gamma}\Psi_{\phi}(X,Y,\phi) = h_2 u \tag{19}$$

$$\frac{1}{\gamma}\Phi_{\phi}(X,Y,\phi) = h_2 v \tag{20}$$

Again from (17), (18), (19) and (20) we have

$$\Psi_{\phi}(X,Y,\phi) = \gamma h_2 UL\overline{f}_{\phi} \tag{21}$$

$$\Phi_{\phi}(X,Y,\phi) = \gamma h_1 VM\overline{g}_{\phi} \tag{22}$$

In view of equation (17) and (18) the velocity components u and v may be found as

$$u = UL\overline{f_{\phi}} \tag{23}$$

and

$$v = VM\overline{g_{\phi}} \tag{24}$$

Here the subscripts denote the partial differentiation with respect to corresponding arguments. Since the external velocity U_e and V_e is independent of ς, they are also be independent of ϕ, yielding $(U_e)_\phi = 0$; $(V_e)_\phi = 0$. Form (16), using (13), (21) and (22) we have

$$h_1 h_2 w = -[\Psi_X + \Phi_Y] + \phi\gamma_X h_2 LU\bar{f}_\phi + \phi\gamma_Y h_1 MV\bar{g}_\phi \tag{25}$$

The convective operator

$$h_2 u\frac{\partial}{\partial\xi} + h_1 v\frac{\partial}{\partial\eta} + h_1 h_2 w\frac{\partial}{\partial\varsigma} \tag{26}$$

In terms of new set of variables (X,Y,ϕ) may be derived. Using (13), (25), (21) and (22), equation (26)

$$h_2UL\bar{f}_\phi\left[\frac{\partial}{\partial X} - \frac{\phi}{\gamma}\gamma_X\frac{\partial}{\partial\phi}\right] + h_1MV\bar{g}_\phi\left[\frac{\partial}{\partial Y} - \frac{\phi}{\gamma}\gamma_Y\frac{\partial}{\partial\phi}\right] - [\Psi_X + \Phi_Y]\frac{1}{\gamma}\frac{\partial}{\partial\phi}$$

$$+\left[\phi\gamma_X h_2 LU\bar{f}_\phi + \phi\gamma_Y h_1 MV\bar{g}_\phi\right]\frac{1}{\gamma}\frac{\partial}{\partial\phi} = h_2UL\bar{f}_\phi\frac{\partial}{\partial X} + h_1MV\bar{g}_g\frac{\partial}{\partial Y}$$

$$\left[\frac{\Psi_X}{\gamma} + \frac{\Phi_Y}{\gamma}\right]\frac{\partial}{\partial\phi} \tag{27}$$

By virtue of equation (27), equations (11) and (12) transformed to U - momentum and V - momentum equations be

U - momentum :

$$\upsilon\bar{f}_{\phi\phi\phi} + \frac{\gamma(\gamma h_2UL)_X}{h_1h_2}\bar{f}\bar{f}_{\phi\phi} + \frac{\gamma(\gamma h_1VM)_Y}{h_1h_2}\bar{g}\bar{f}_{\phi\phi} - \frac{(UL)_X\gamma^2}{h_1}\bar{f}_\phi^2 -$$

$$\frac{MV\gamma^2}{h_2}\left[\frac{(UL)_Y}{UL} + \frac{h_{1Y}}{h_1}\right]\bar{f}_\phi\bar{g}_\phi + \frac{V^2M^2h_{2X}\gamma^2}{h_1h_2UL}\bar{g}_\phi^2 + \frac{U_eU_{eX}\gamma^2}{h_1UL} +$$

$$\frac{V_eU_{eY}\gamma^2}{h_2UL} + \frac{U_eV_eh_{1Y}\gamma^2}{h_1h_2UL} - \frac{V^2h_{2X}\gamma^2}{h_1h_2UL} = 0 \tag{28}$$

and V - momentum :

$$\upsilon \bar{g}_{\phi\phi\phi} + \frac{\gamma(\gamma h_2 UL)_X}{h_1 h_2} \bar{f}\bar{g}_{\phi\phi} + \frac{\gamma(\gamma h_1 VM)_Y}{h_1 h_2} \bar{g}\bar{g}_{\phi\phi} - \frac{(VM)_Y \gamma^2}{h_2} \bar{g}_{\phi}^2 -$$

$$\frac{UL\gamma^2}{h_1}\left[\frac{(VM)_X}{VM} + \frac{h_{2X}}{h_2}\right]\bar{f}_{\phi}\bar{g}_{\phi} + \frac{U^2 L^2 h_{1Y}\gamma^2}{h_1 h_2 VM}\bar{f}_{\phi}^2 + \frac{U_e V_{eX}\gamma^2}{h_1 MV} +$$

$$\frac{V_e V_{eY}\gamma^2}{h_2 MV} + \frac{U_e V_e h_{2X}\gamma^2}{h_1 h_2 VM} - \frac{U^2 h_{1Y}\gamma^2}{h_1 h_2 VM} = 0 \qquad (29)$$

Here the boundary conditions which must be imposed to determine the solutions of the equations (28) and (29) are

(i) The fluid must adhere to the transformed surface and this must be a streamline.

However, if the developable surface be impervious, that is

$$\bar{f}_{\phi}(0) = \bar{g}_{\phi}(0) = 0$$

(ii) The fluid a large distance form the said surface and must be undisturbed by the presence of the boundary layer

$$\bar{f}_{\phi}(\infty) = \bar{g}_{\phi}(\infty) = 1$$

If the general boundary conditions (i) and (ii) be introduced, without loss of generally we may write $UL = U_e$ and $VM = V_e$, than the equations (28) and (29) becomes

U - momentum :

$$\upsilon \bar{f}_{\phi\phi\phi} + a_1 \bar{f}\bar{f}_{\phi\phi} + a_2 \bar{g}\bar{f}_{\phi\phi} + a_3\left(1 - \bar{f}_{\phi}^2\right) + a_4\left(1 - \bar{f}_{\phi}\bar{g}_{\phi}\right) - a_5\left(1 - \bar{g}_{\phi}^2\right) = 0 \quad (30)$$

and V - momentum :

$$\upsilon \bar{g}_{\phi\phi\phi} + a_1 \bar{f}\bar{g}_{\phi\phi} + a_2 \bar{g}\bar{g}_{\phi\phi} + a_6\left(1 - \bar{g}_{\phi}^2\right) + a_7\left(1 - \bar{f}_{\phi}\bar{g}_{\phi}\right) - a_8\left(1 - \bar{f}_{\phi}^2\right) = 0 \quad (31)$$

where $a_1, a_2, a_3, \ldots\ldots, a_8$ are contractions of the function of X, Y, given by

$$(i)\quad \frac{\gamma(\gamma h_2 U_e)_X}{h_1 h_2} = a_1$$

$$(ii)\quad \frac{\gamma(\gamma h_1 V_e)_Y}{h_1 h_2} = a_2$$

$$(iii)\quad \frac{\gamma^2 (U_e)_X}{h_1} = a_3$$

$$(iv)\quad \frac{\gamma^2}{h_1 h_2}\frac{V_e}{U_e}(h_1 U_e)_Y = a_4$$

$$(v)\quad \frac{\gamma^2}{h_1 h_2}\frac{V_e^2}{U_e} h_{2X} = a_5$$

$$(vi)\quad \frac{\gamma^2}{h_1 h_2} V_{eY} = a_6$$

$$(vii)\quad \frac{\gamma^2}{h_1 h_2}\frac{U_e}{V_e}(h_2 V_e)_X = a_7$$

$$(viii)\quad \frac{\gamma^2}{h_1 h_2}\frac{U_e^2}{V_e} h_{1Y} = a_8 \qquad (32)$$

Similar solutions for (30) and (31) exist only when will all the a's are finite and independent of X,Y. Thus, (30) and (31) will becomes non-linear ordinary differential equations. These eight diffetential equations determine the forms U_e, V_e, h_1, h_2, and γ. Two such rather obvious assumptions are that the velocities U_e and V_e are propotional to each other and the flow is incompressible.

Conclusion : Hence the relations stated by equations (32) may be treated to be the conditions which furnish us equations for $U_e(X,Y)$, $V_e(X,Y)$ and $\gamma(X,Y)$ the scalar factors for the velocity component u,v and the ordinate Y for arbitrary choices of h_2 and h_3. These scale factors will uniquely determine the flow characteristics of the boundary layer.

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