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# EXISTENCE OF GLOBAL SOLUTIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

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## ABSTRACT

The aim of this paper is to establish the existence of solution of certain nonliear integrodifferential equations in Banach spaces. The results are established using the method of analytic semigroup and the contraction mapping principle.

***Key Words:*** *Existence of solutions, analytic semigroup, fixed point theorem.*

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## 1. INTRODUCTION

Dhakne and Lamb [3] studied the existence of global solutions of nonlinear integrodifferential equations of the form

$$x'(t)+Ax(t)=f\left(t,x(t),\int_{t_0}^{t} a(t,s)g(s,x(s))ds\right) \quad t\in[t_0,t_1] \tag{1}$$

$$x(t_0) \;=x_0 \tag{2}$$

where - A generates a strongly continuous semigroup of bounded linear operators, $a:[t_0,t_1]\times[t_0,t_1]\to R$, $f:[t_0,t_1]\times X\times X\to X$, $g:[t_0,t_1]\times X\to X$ in a Banach space $X$ and $x_0\in X$. The results are obtained by using the Leray-Schauder Alternative. In this paper we try to establish the existence of global solutions of the above nonlinear integrodifferential equations in which - A is the infinitesimal generator of an analytic semigroup and the results are established by applying contraction mapping principle. The problem of existence, uniqueness and other properties of the special forms of the equation (1) has been studied by many authors using different techniques [1-5,7].

## 2. PRELIMINARIES

Let - A be the infinitesimal generator of an analytic semigroup $T(t)$, on the Banach space $X$. The operator $A^{\alpha}$ can be defined for $0\le\alpha\le 1$ as the inverse of the bounded linear operator

$$A^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}T(t)dt$$

and $A^{\alpha}$ is a closed linear invertible operator with domain $D(A^{\alpha})$ dense in $X$. The closedness of $A^{\alpha}$ implies that $D(A^{\alpha})$ endowed with the graph norm of $A^{\alpha}$, that is $|||x|||=\|x\|+\|A^{\alpha}x\|$, is a Banach space. Since $A^{\alpha}$ is invertible its graph norm $|||\cdot|||$ is equivalent to the norm $\|x\|_{\alpha}=\|A^{\alpha}x\|$. Thus, $D(A^{\alpha})$ equipped with the norm $\|\cdot\|_{\alpha}$ is a Banach space which we denote by $X_{\alpha}$. From this definition it is clear that $0<\alpha<\beta$ implies $X_{\alpha}\supset X_{\beta}$ and that the imbedding of $X_{\beta}$ in $X_{\alpha}$ is continuous. For more results on fractional power operator one can refer [6].

**Assumption (F) :** Let U be an open subset of $R^{+}\times X_{\alpha}\times X_{\alpha}$. The function $f:U\to X$ satisfies the condition : if for every $(t,x,y)\in U$ there is a neighbourhood $V\subset U$ and constants $L\ge 0$, $0<\nu\le 1$ such that

$$\|f(t_1,x_1,y_1)-f(t_2,x_2,y_2)\|\le L\left(|t_1-t_2|^{\nu}+\|x_1-x_2\|_{\alpha}+\|y_1-y_2\|_{\alpha}\right) \tag{3}$$

for all $(t_i,x_i,y_i)\in V$, $i=1,2$.

Further we assume the following conditions :

(*i*) $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $T(t)$, satisfying $\|T(t)\| \leq M$, where $M > 0$ is a constant, $0 \in \rho(-A)$.

(*ii*) $a : [t_0,t_1] \times [t_0,t_1] \to R$, is Holder continuous in $t$, $s$ and there exist constants $C_1>0$, $C_2>0$ and $0<\theta\leq 1$, such that $|a(t,s)| \leq C_1$ and $|a(t,\tau) - a(s,\tau)| \leq C_2|t - s|^{\theta}$.

(*iii*) $g : [t_0,t_1] \times X \to X$ is continuous and there exists a constant $C_3>0$, $C_4>0$ such that $\|g(t,x_1)-g(t,x_2)\| \leq C_3\|x_1-x_2\|_{\alpha}$ and $\|g(t,x)\| \leq C_4$.

## 3. Main Results

**Theorem 3.1 :** If $0 < \alpha < 1$ and $f$ satisfies the assumption ($F$) and ($i$) - ($iii$) then for every initial data $(t_0, x_0, y_0) \in U$, the integrodifferential equation (1) has a unique local solution $u \in C([t_0,t_1) : X) \cap C^1((t_0,t_1) : X)$.

**Proof :** From our assumptions on the operator $A$ it follows that $\|A^{\alpha}T(t)\| \leq C_{\alpha}t^{-\alpha}$ for $t > 0$. For the rest of the proof, we fix $(t_0,x_0,y_0) \in U$ and choose $t_1' > t_0, \delta > 0$ such that the estimate (3) with some fixed constants $L$ and $\nu$ holds in the set $V = \{(t,x,y) : t_0 \leq t \leq t_1', \|x-x_0\|_{\alpha} \leq \delta\}$. Let $M_1 = max \|f(t,x_0,0)\|$ and $M_2 = max \|g(t,x_0)\|$ and choose '$t_1$' such that

$$\left\|T(t-t_0)A^{\alpha}x_0 - A^{\alpha}x_0\right\| < \frac{\delta}{2} \text{ for } t_0 \leq t \leq t_1 \quad \text{and}$$

$$0 < t_1 - t_0 < \min\left\{ t_1' - t_0, \left[ \frac{\delta}{2} C_{\alpha}^{-1}(1-\alpha)L^{-1}\delta^{-1} \right.\right.$$

$$\left.\left. \times(1+C_1C_3)^{-1}\left[1+\frac{LM_2C_1+M_1}{L\delta(1+C_1C_3)}\right]^{-1}\right]^{\frac{1}{1-\alpha}} \right\}.$$

Let $Y$ be the Banach space $C([t_0,t_1] : X)$ with the supremum norm which we denote by $\|.\|_Y$. Define $F : Y \to Y$ by

$$F(y(t)) = T(t-t_0)A^{\alpha}x_0 + \int_{t_0}^{t} A^{\alpha}T(t-s)f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau))d\tau\right)ds\Big).$$

Clearly $F$ maps $Y$ into itself and for every $y \in Y$, $Fy(t_0) = A^{\alpha}x_0$.

Let $S$ be the non-empty closed and bounded subset of $Y$ defined by

$$S = \{y \in Y : y(t_0) = A^{\alpha}x_0, \|y(t) - A^{\alpha}x_0\| \leq \delta\}.$$

For $y \in S$, we have

$$\begin{aligned}
&\left\|Fy(t) - A^{\alpha}x_0\right\| \\
&\leq \left\|T(t-t_0)A^{\alpha}x_0 - A^{\alpha}x_0\right\| \\
&+ \left\|\int_{t_0}^{t} A^{\alpha}T(t-s)\left[f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau))d\tau\right) - f(s, x_0, 0)\right]ds\right\| \\
&+ \left\|\int_{t_0}^{t} A^{\alpha}T(t-s)f(s, x_0, 0)ds\right\| \\
&\leq \frac{\delta}{2} + \left\|\int_{t_0}^{t} A^{\alpha}T(t-s)\right\|\left[L\left(\left\|y(s) - A^{\alpha}x_0\right\| + \left\|\int_{t_0}^{s} a(s,\tau)\left[g(\tau, A^{-\alpha}y(\tau))\right.\right.\right.\right. \\
&\left.\left.\left.\left. - g(\tau, x_0) + g(\tau, x_0)\right)\right]d\tau\right\|\right)\right]ds + M_1\left\|\int_{t_0}^{t} A^{\alpha}T(t-s)ds\right\| \\
&\leq \frac{\delta}{2} + \int_{t_0}^{t} \left\|A^{\alpha}T(t-s)\right\|\left[L\delta + LC_1C_3\delta + LC_1M_2 + M_1\right]ds \\
&\leq \frac{\delta}{2} + C_{\alpha}(1-\alpha)^{-1}(t_1-t_0)^{1-\alpha}\left[L\delta(1 + C_1C_3) + LC_1M_2 + M_1\right] \\
&\leq \frac{\delta}{2} + C_{\alpha}(1-\alpha)^{-1}(t_1-t_0)^{1-\alpha}L\delta(1 + C_1C_3)\left[1 + \frac{LM_2C_1 + M_1}{L\delta(1 + C_1C_3)}\right] \\
&\leq \delta.
\end{aligned}$$

Therefore, $F: S \to S$. Furthermore, if $y_1, y_2 \in S$ then

$$\|Fy_1(t) - Fy_2(t)\|$$

$$\leq \left\| \int_{t_0}^{t} A^{\alpha} T(t-s) ds \right\| \left\| \left[ f\left(s, A^{-\alpha} y_1(s), \int_{t_0}^{s} a(s,\tau) g\left(\tau, A^{-\alpha} y_1(\tau)\right) d\tau\right) \right. \right.$$

$$\left.\left. - f\left(s, A^{-\alpha} y_2(s), \int_{t_0}^{s} a(s,\tau) g\left(\tau, A^{-\alpha} y_2(\tau)\right) d\tau\right)\right] \right\|$$

$$\leq \left\| \int_{t_0}^{t} A^{\alpha} T(t-s) ds \right\| L\left(\|y_1 - y_2\|_Y + (C_1 C_3)\|y_1 - y_2\|_Y\right)$$

$$\leq C_{\alpha}(1-\alpha)^{-1}(t_1 - t_0)^{1-\alpha} L \|y_1 - y_2\|_Y \left(1 + (C_1 C_3)\right)$$

$$\leq \frac{1}{2}\left[1 + \left[\frac{LM_2 C_1 + M_1}{\delta(1 + C_1 C_3)}\right]\right]^{-1} \|y_1 - y_2\|_Y$$

$$\leq \frac{1}{2}\|y_1 - y_2\|_Y$$

which implies

$$\|Fy_1 - Fy_2\| \leq \frac{1}{2}\|y_1 - y_2\|_Y .$$

By the contraction mapping theorem the mapping F has a unique fixed point $y \in S$. This fixed point satisfies the integral equation

$$y(t) = T(t-t_0) A^{\alpha} x_0 + \int_{t_0}^{t} A^{\alpha} T(t-s) f\left(s, A^{-\alpha} y(s), \int_{t_0}^{s} a(s,\tau) g\left(\tau, A^{-\alpha} y(\tau)\right) d\tau \right) ds \qquad (4)$$

for $t_0 \leq t \leq t_1$. From (3) and the continuity of $y$ it follows that

$$t \to f\left(t, A^{-\alpha} y(t), \int_{t_0}^{t} a(t,s) g\left(s, A^{-\alpha} y(s)\right) ds\right)$$

is continuous on $[t_0,t_1]$ and a fortiori bounded on this interval. Let

$$\left\| f\left(t, A^{-\alpha}y(t), \int_{t_0}^{t} a(t,s)g(s, A^{-\alpha}y(s))ds\right) \right\| \le N, \; for \; t_0 \le t \le t_1 . \quad (5)$$

Next we want to show that

$$t \to f\left(t, A^{-\alpha}y(t), \int_{t_0}^{t} a(t,s)g(s, A^{-\alpha}y(s))ds\right)$$

is locally Holder continuous on $(t_0, t_1]$. First we show that the solution $y$ of (4) is locally Holder continuous on .$(t_0,t_1]$. Note that for every $\beta$ satisfying $0<\beta<1-\alpha$ and every $0 < h < 1$,we have

$$\|(T(h)-I)A^{\alpha}T(t-s)\| \le C_{\beta}h^{\beta}\|A^{\alpha+\beta}T(t-s)\| \le Ch^{\beta}(t-s)^{-(\alpha+\beta)} .$$

If $t_0 < t < t+h \le t_1$, then

$$\|y(t+h)-y(t)\|$$

$$\le \|(T(h)-I)A^{\alpha}T(t-t_0)x_0\|$$

$$+\int_{t_0}^{t} \left\|(T(h)-I)A^{\alpha}T(t-s)f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau))d\tau\right)\right\| ds \Big)$$

$$+\int_{t_0}^{t+h} \left\|A^{\alpha}T(t+h-s)A^{\alpha}T(t-s)f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau))d\tau\right)\right\| ds \Big)$$

$$= I_1+I_2+I_3. \quad (6)$$

Using (4) and (5) we estimate each of the terms of (6) separately as

$$I_1 \le C(t-t_0)^{-(\alpha+\beta)}h^{\beta}\|x_0\| \le M_3h^{\beta} \quad (7)$$

$$I_2 \le CNh^{\beta}\int_{t_0}^{t} (t-s)^{-(\alpha+\beta)}ds \le M_4h^{\beta} \quad (8)$$

$$I_3 \le NC_{\alpha}\int_{t}^{t+h} (t+h-s)^{-\alpha} = \frac{NC_{\alpha}}{1-\alpha}h^{1-\alpha} \le M_5h^{\beta} \quad (9)$$

Note that $M_4$ and $M_5$ can be chosen to be independent of $t \in [t_0, t_1]$ while $M_3$ depends on $t$ and blows up at $t \downarrow t_0$. Combining (7), (8) and (9) with these estimates it follows that for every $t_0' > t_0$, there exists a constant $M_6$ such that

$$\|y(t) - y(s)\| \le M_6 \left(|t-s|^{\beta}\right) \, for \; t_0 \le t_0' \le t, \quad s \le t_1$$

and therefore $y$ is locally Holder continuous on $(t_0, t_1]$. The local Holder continuity of

$$t \to f\left(t, A^{-\alpha} y(t), \int_{t_0}^{t} a(t,s) g\left(s, A^{-\alpha} y(s)\right) ds\right)$$

follows now from

$$\left\| f\left(t, A^{-\alpha} y(t), \int_{t_0}^{t} a(t,\tau) g\left(\tau, A^{-\alpha} y(\tau)\right) d\tau\right) - f\left(s, A^{-\alpha} y(s), \int_{t_0}^{s} a(s,r) g\left(r, A^{-\alpha} y(r)\right) dr\right)\right\|$$

$$\le L\left(|t-s|^{\nu} + \|y(t) - y(s)\| + \left\|\int_{t_0}^{t} [a(t,\tau) - a(s,\tau)] g\left(\tau, A^{-\alpha} y(\tau)\right) d\tau\right\| + \left\|\int_{t_0}^{t} a(s,\tau) g\left(\tau, A^{-\alpha} y(\tau)\right) d\tau - \int_{t_0}^{s} a(s,r) g\left(r, A^{-\alpha} y(r)\right) dr\right\|\right.$$

$$\le L\left(|t-s|^{\nu} + M_6 |t-s|^{\beta} + C_1 C_4 |t-s|^{\theta} (t_1 - t_0) + C_1 C_4 (t-s)\right)$$

$$\le L\left(|t-s|^{\nu} + M_6 |t-s|^{\beta} + |t-s|^{\eta} [C_2 C_4 (t_1 - t_0) + C_1 C_4]\right), \quad 0 < \eta \le 1$$

$$\le K\left(|t-s|^{\nu} + |t-s|^{\beta} + |t-s|^{\eta}\right)$$

for some positive constant $K > 0$.

Let $y$ be the solution of (4) and consider the inhomogeneous initial value problem

$$x'(t) + Ax = f\left(t, A^{-\alpha}y(t), \int_{t_0}^{t} a(t,s)g(s, A^{-\alpha}y(s))ds\right) \quad t \in [t_0, t_1] \tag{10}$$

$$x(t_0) = x_0.$$

By Corollary 4.3.3. of [6] this problem has a unique solution of $x \in C^1((t_0, t_1] : X)$ is given by

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t-s) f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau)d\tau)\right)ds\Big). \tag{11}$$

For $t > t_0$ each term of (11) is in $D(A)$ and a fortiori in $D(A^{\alpha})$ we find the given equation becomes

$$A^{\alpha}x(t) = T(t - t_0)A^{\alpha}x_0$$

$$+ \int_{t_0}^{t} A^{\alpha}T(t-s) f\left(s, A^{-\alpha}y(s), \int_{t_0}^{s} a(s,\tau)g(\tau, A^{-\alpha}y(\tau)d\tau)\right)ds. \tag{12}$$

But by (4) the right hand side of (12) equals $y(t)$ and therefore $x(t) = A^{-\alpha}y(t)$ and by (11), $x$ is a $C^1((t_0, t_1] : X)$ solution of (1) - (2). The uniqueness of $x$ follows from the uniqueness of the solutions of (4) and (10).

Now we establish the global solution of (1) - (2).

**Theorem 3.2 :** Let $0 \in \rho$ $(-A)$ and let $-A$ be the infinitesimal generator of an analytic semigroup T(t) satisfying $\|T(t)\| \leq M$ for $t \geq 0$. Let $f$: $[0,\infty) \times X_\alpha \times X_\alpha \to X$ satisfy $(F)$. If there is a continuous nondecreasing real valued function $k(t)$ such that

$$\left\| f\left(t, x, \int_0^t a(t,s)g(s, A^{-\alpha}y(s))ds\right) \right\| \leq k(t)(1 + \|x\|_\alpha) \quad for \quad t \geq 0, x \in X_\alpha$$

then for every $(x_0, y_0) \in X_\alpha$, the problem (1) has a unique solution $x$ which exists for all $t \geq 0$.

**Proof :** Applying Theorem 3.1 we can continue the solution of (1) as long as $\|x(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if $x$ exists on [0,b) then $\|x(t)\|_\alpha$ is bounded as $t\uparrow b$. Since

$$A^\alpha x(t) = A^\alpha T(t)x_0 + \int_0^t A^\alpha T(t-s)f\left(s, x(s), \int_0^s a(s,\tau)g(\tau, x(\tau))d\tau\right)ds$$

it follows that

$$\|x(t)\|_\alpha \le M\|A^\alpha x_0\| + \frac{k(b)C_\alpha b^{1-\alpha}}{1-\alpha} + k(b)C_\alpha \int_0^t (t-s)^{-\alpha}\left[\|x(t)\|_\alpha + 1\right]ds$$

which implies that $\|x(t)\|_\alpha \le C$ on $[0,b)$. Hence the proof.

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