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FUNCTIONS WITH $\pi g\alpha$ - CLOSED GRAPHS

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Abstract

In this paper the concept of $\pi g\alpha$ - closed graphs for functions between topological spaces are introduced with the help of $\pi g\alpha$ - open sets. Some basic properties of functions with a $\pi g\alpha$ - closed graph have been obtained.

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1. Introduction

In 1969, Long [8] studied the properties of functions with closed graph in great detail. Closed graph notion is now an active area of research and a large number of topolgists have established its far - reaching effect on different concepts of point set topology. In 1983, Dube et. al., [3] introduced the notion of semi-closed graph utilizing semi-open sets introduced by Levine [7].

In this paper we introduce $\pi g\alpha$-closed graphs with the aid of $\pi g\alpha$-open sets.

2. Preliminaries

Throughout (X,τ), (Y, σ) (or simply X, Y) will always denote topological spaces on which no separations axioms are assumed unless explicitly stated. If A is a subset of a space (X, τ) then the closure of A (resp. interior of A) is denoted by $cl(A)$ (resp. int A). A subset A is said to be regular open (resp. regular closed) if $A = \text{intcl}(A)$ (resp. $A = \text{clint}(A)$). The finite union of regular open sets is said to be π-open. α-closure of A is the intersection of all α-closed sets containing A. It is well known that $\alpha clA = A \cup cl(\text{int}(cl(A)))$.

Definition 2.1 : A subset A of (X, τ) is called

(a) α-closed [9] if clintclA $\subset$ A

(b) $\pi g\alpha$-closed [1] of $\alpha clA \subset U$ whenever $A \subset U$ and U is π-open

(c) $\pi g\alpha$-open [1] if X-A is $\pi g\alpha$-closed

(d) $\pi g\alpha$-clopen [1] if A is both $\pi g\alpha$-open and $\pi g\alpha$-closed

The family of all $\pi g\alpha$-open sets containing x (resp. $\pi g\alpha$-closed sets) is denoted by $\pi G\alpha O(X, x)$ (resp. $\pi G\alpha C(X,x)$).

Definition 2.2: A function $f: (X,\tau) \to (Y, \sigma)$ is called

(a) $\pi g\alpha$-continuous [5] (resp. $\pi g\alpha$-irresolute) if $f^{-1}(V)$ is $\pi g\alpha$-open in X for each open set V (resp. $\pi g\alpha$-open) in Y.

(b) M-$\pi g\alpha$-open [5] if $f(U)$ is $\pi g\alpha$-open for all $U \in \pi G\alpha O(X)$

(c) contra-$\pi g\alpha$-continuous [2] if $f^{-1}(V)$ is $\pi g\alpha$-open in X for each closed set V of Y.

Definition 2.3 : [4] Let $f: (X,\tau) \to (Y, \sigma)$ be any function. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f.

Definition 2.4 : [4] Let X, Y be topological spaces. A mapping $f: X \to Y$ is said to have a closed graph if its graph $G(f)$ is closed in the product space $X \times Y$.

Lemma 2.5 : [4] Let $f: X \to Y$ be given. Then $G(f)$ is closed iff for each $(x, y) \in X \times Y - G(f)$ there exists $U \in \Sigma(x)$ in X and $V \in \Sigma(y)$ in Y such that $f(U) \cap V = \phi$.

Lemma 2.6: [6] Let f : $X \to Y$ be given. Then $G(f)$ is α-closed iff for each $(x, y) \in X \times Y - G(f)$ there exist $U \in \alpha O(X, x)$ and $V \in \alpha O(Y, y)$ in Y such that $f(U) \cap V = \phi$.

Definition 2.7 : [2] A space X is called

(a) $\pi g\alpha$-T_1 if for $x, y \in X$ such that $x \neq y$ there exist a $\pi g\alpha$-open set containing x but not y and a $\pi g\alpha$ - open set containing y but not x.

(b) $\pi g\alpha$-T_2 [2] if for $x, y \in X$ such that $x \neq y$ there exist $U\in \pi G\alpha O(X,x)$ $V\in \pi G\alpha O(Y,y)$ such that $U\cap V=\phi$.

(c) $\pi G\alpha O$-compact [5] if every $\pi g\alpha$-open cover of X admits a finite subcover.

(d) $\pi g\alpha$-connected [5] if X cannot be expressed as the disjoint union of two $\pi g\alpha$-open sets.

Lemma 2.8 [5] : Every $\pi g\alpha$-closed subset of a $\pi G\alpha O$- compact space is $\pi G\alpha O$-compact relative to X.

3. $\pi g\alpha$-Closed Graphs

Definition 3.1 : For a function $f : X \rightarrow Y$, the graph G(f) is said to be $\pi g\alpha$-closed graph if for each $(x, y) \in X\times Y - G(f)$ there exist $U\in \pi G\alpha O(X, x)$, $V\in \pi G\alpha O(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.

Lemma 3.2 : The function $f: X\rightarrow Y$ has a $\pi g\alpha$-closed graph iff for each $(x, y)\in X\times Y - G(f)$ there exist $U\in \pi G\alpha O(X, x)$, $V\in \pi G\alpha O(Y, y)$ such that $f(U)\cap V = \phi$.

Proof : It follows from definition and the fact that for any subsets $U\subset X$ and $V\subset Y$, $(U\times V)\cap G(f) = \phi$ iff $f(U)\cap V = \phi$.

Theorem 3.3 :

(a) Every closed graph is $\pi g\alpha$-closed graph.

(b) Every α-closed graph is $\pi g\alpha$-closed graph.

Proof : Straight forward.

Converses of the above are not true as seen in the following example.

Example 3.4 : Let $X = \{a, b\}$ $Y = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}\}$ and $\sigma = \{\phi, \{c,d\}, Y\}$ respectively. Let $f : X \to Y$ be the mapping defined by $f(a)=a$, $f(b) = b$. Then $G(f)$ is $\pi g\alpha$-closed but not closed.

Example 3.5 : In example 3.4, $G(f)$ is $\pi g\alpha$-closed but not α-closed.

Remark 3.6 : Functions having $\pi g\alpha$-closed graph need not be $\pi g\alpha$-continuous.

Example 3.7 : Let $X = \{a,b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and σ = discrete topology.

Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then $G(f)$ is $\pi g\alpha$-closed but f is not $\pi g\alpha$-continuous.

Remark 3.8 : A $\pi g\alpha$-continuous function need not have a $\pi g\alpha$-closed graph as shown by the following example.

Example 3.9 : Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{\phi, X, \{a,b\}\}$ and f : $(X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is $\pi g\alpha$-continuous but $G(f)$ is not $\pi g\alpha$-closed.

Remark 3.10 : Example 3.7, 3.9 show that $\pi g\alpha$-closed graph and $\pi g\alpha$-continuous functions are independent concepts.

Theorem 3.11 : Let $f : X \to Y$ be $\pi g\alpha$-irresolute surjection where X is an arbitrary topological space and Y is $\pi g\alpha$-T_2. Then $G(f)$ is $\pi g\alpha$-closed.

Proof : Let $(x, y) \in X{\times}Y\text{-}G(f)$. Then $y \neq f(x)$. Since Y is $\pi g\alpha$-T_2 there exist $\pi g\alpha$-open sets U, V in Y such that $f(x) \in U$, $y \in V$ and $U \cap V = \phi$. Since f is $\pi g\alpha$-irresolute, $W = f^{1}(U) \in \pi G\alpha O(X, x)$. Hence $f(W) = f(f^{1}(U)) \subset U$. This implies $f(W) \cap V = \phi$. Hence by lemma 3.2, $G(f)$ is $\pi g\alpha$-closed.

Theorem 3.12 : Let $X \to Y$ be $\pi g\alpha$-continuous surjection where X is an arbitrary topological space and Y is T_2. Then $G(f)$ is $\pi g\alpha$-closed.

Proof : Let $(x, y) \in X{\times}Y\text{-}G(f)$. Then $y \neq f(x)$. Since Y is T_2 there exist open sets U and V containing $f(x)$ and y respectively such that $U \cap V = \phi$. Since f is $\pi g\alpha$-contiuous,

$f^{-1}(U) = W \in \pi G\alpha O(X,x)$. Since f is surjection, $f(W) = f(f^{-1}(U)) \subset U$. Hence $f(W) \cap V = \phi$. By lemma 3.2, $G(f)$ is $\pi g\alpha$-closed.

Remark 3.13 : From example 3.9 we find that the condition $\pi g\alpha$-irresolute in theorem 3.11 cannot be replaced by $\pi g\alpha$-continuous.

Theorem 3.14 : Let $f : X \to Y$ be any surjection with $G(f)$ $\pi g\alpha$-closed. Then Y is $\pi g\alpha$-T_1.

Proof : Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is surjective there exist $x_1 \in X$ such that $f(x_1) = y_2$. Now $(x_1, y_1) \in X \times Y\text{-}G(f)$. Since $G(f)$ is $\pi g\alpha$-closed, there exist a $\pi g\alpha$-open set U_1 containing x_1 and a $\pi g\alpha$-open set V_1 contianing y_1 such that $f(U_1) \cap V_1 = \phi$. Now $x_1 \in U_1 \Rightarrow f(x_1) = y_2 \in f(U_1)$. $y_2 \in f(U_1)$ and $f(U_1) \cap V_1 = \phi \Rightarrow y_2 \notin V_1$. Again, since f is surjective, there exist a point $x_2 \in X$ such that $f(x_2) = y_1$. Now $(x_2, y_2) \in X \times Y\text{-}G(f)$. Since $G(f)$ is $\pi g\alpha$-closed, there exist $U_2 \in \pi G\alpha O(X, x_2)$ and $V_2 \in \pi G\alpha O(Y, y_2)$ such that $f(U_2) \cap V_2 = \phi$. Now $x_2 \in U_2 \Rightarrow f(x_2) = y_1 \in f(U_2)$. Now $y_1 \in f(U_2)$ and $f(U_2) \cap V_2 = \phi$ $\Rightarrow y_1 \notin V_2$. Thus we obtain sets V_1, $V_2 \in \pi G\alpha O(Y)$ such that $y_1 \in V_1$ but $y_2 \notin V_1$ while $y_2 \in V_2$, $y_1 \notin V_2$. Hence Y is $\pi g\alpha$-T_1.

Theorem 3.15 : Let $f : X \to Y$ be any M-$\pi g\alpha$-open surjection with $G(f)$ $\pi g\alpha$-closed. Then Y is $\pi g\alpha$-T_2.

Proof : Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is surjective, there exist $x_1 \in X$ such that $f(x_1) = y_2$. Then $(x_1, y_1) \in X \times Y\text{-}G(f)$. Since $G(f)$ is $\pi g\alpha$-closed by lemma 3.2, there exist $U \in \pi G\alpha O(X, x_1)$ and $V \in \pi G\alpha O(Y, y_1)$ such that $f(U) \cap V = \phi$. Since f is M-$\pi g\alpha$-open, $f(U)$ is $\pi g\alpha$-open in Y. Now $x_1 \in U \Rightarrow f(x_1) = y_2 \in f(U)$. Therefore, there exist $V \in \pi G\alpha O(Y, y_1)$ and $f(U) \in \pi G\alpha O(Y, y_2)$ such that $f(U) \cap V = \phi$. Hence Y is $\pi g\alpha$-T_2 space.

Theorem 3.16 : Let $f : X \to Y$ be injective with $G(f)$ $\pi g\alpha$-closed. Then X is $\pi g\alpha$-T_1.

Proof : Let $x_1, x_2 (\neq x_1) \in X$. Since f is injective, $f(x_1) \neq f(x_2)$.

Hence $(x_1, f(x_2)) \in X \times Y\text{-}G(f)$. Since $G(f)$ is $\pi g\alpha$-closed by lemma 3.2 there exist $U \in \pi G\alpha O(X, x_1)$ and $V \in \pi G\alpha O(Y, f(x_2))$ such that $f(U) \cap V = \phi$. $f(x_2) \in V$ and

$f(U)\cap V=\phi \Rightarrow f(x_2)\notin f(U)$ and so $x_2\notin U$. Similarly for $(x_2, f(x_1))\in X\times Y\text{-}G(f)$ there exist $U\in \pi G\alpha O(X, x_2)$ and such $V_1\in \pi G\alpha O(Y, f(x_1))$ $f(U_1)\cap V_1 = \phi$. Therefore $f(x_1)\notin f(U_1)$ and so $x_1\notin U_1$. Hence we obtain $\pi g\alpha$-open sets U and U_1 in X respectively such that $x_1\in U$ but $x_2\notin U$ and $x_2\in U_1$ but $x_1\notin U_1$. Thus X is $\pi g\alpha\text{-}T_1$.

Corollary 3.17 : Let $f: X\to Y$ be bijective and $G(f)$ be $\pi g\alpha$-closed. Then both X and Y are $\pi g\alpha\text{-}T_1$.

Proof : It follows from theorem 3.14 and 3.16.

Theorem 3.18 : If $f: X\to Y$ is injective, $\pi g\alpha$-irresolute with a $\pi g\alpha$-closed graph then X is $\pi g\alpha\text{-}T_2$.

Proof : Let $x_1, x_2\ (\neq x_1)\in X$. Since f is injective, $f(x_1)\neq f(x_2)$.

Hence $(x_1, f(x_2))\in X\times Y\text{-}G(f)$. Since $G(f)$ is $\pi g\alpha$-closed by lemma 3.2 there exist $U\in \pi G\alpha O(X, x_1)$ and $V\in \pi G\alpha O(Y, f(x_2))$ such that $f(U)\cap V=\phi$. Hence $U\cap f^{-1}(V)=\phi$ Since f is $\pi g\alpha$-irresolute, $f^{-1}(V)\in \pi G\alpha O(X, x_2)$. Hence there exist $\pi g\alpha$-open sets U and $f^{-1}(V)$ in X containing x_1 and x_2 respectively such that $U\cap f^{-1}(V) = \phi$. Therefore X is $\pi g\alpha\text{-}T_2$.

Corollary 3.19 : If $f: X\to Y$ is bijective, M-$\pi g\alpha$-open, $\pi g\alpha$-irresolute and $G(f)$ is $\pi g\alpha$-closed then both X and Y are $\pi g\alpha\text{-}T_2$.

Proof : Follows from theorem 3.15 and 3.18.

Definition 3.20 : A function $f: X\to Y$ is sub contra-$\pi g\alpha$-continuous provided there exist an open base B for the topology on Y such that $f^{-1}(V)$ is $\pi g\alpha$-closed in X for every $V\in B$.

Theorem 3.21 : If $f: X\to Y$ is sub contra-$\pi g\alpha$-continuous function and Y is T_1. Then $G(f)$ is $\pi g\alpha$-closed.

Proof : Let $(x, y)\in X\times Y\text{-}G(f)$. Then $y\neq f(x)$. Let B be an open base for the topology on Y. Since f is sub contra-$\pi g\alpha$-continuous, $f^{-1}(V)$ is $\pi g\alpha$-closed in X for every $V\in B$. Since

Y is T_1 there exist a $V \in B$ such that $y \in V$ and $f(x) \notin V$. Then $(x,y) \in (X - f^{-1}(V)) \times V \subset X \times Y - G(f)$. Hence $G(f)$ is $\pi g\alpha$-closed.

Corollary 3.22 : If $f: X \to Y$ contra-$\pi g\alpha$-continuous and Y is T_1 then G(f) is $\pi g\alpha$-closed.

Proof : It follows from the fact that every contra-$\pi g\alpha$-continuous function is sub contra-$\pi g\alpha$-continuous.

4. $\pi G\alpha$-Connectedness

Definition 4.1 : A function $f: X \to Y$ is said to be $\pi g\alpha$-connected if for every $\pi g\alpha$-connected set U, $f(U)$ is $\pi g\alpha$-connected.

Definition 4.2 : A topological space X is locally $\pi G\alpha$-connected if for each $x \in X$ and each and $U \in \pi G\alpha O(X,x)$ there exist a $V \in \pi G\alpha O(X,\tau)$ such that $x \in V \subset U$ where V is $\pi g\alpha$-connected.

Definition 4.3 : Two subsets A and B of a space X are called $\pi g\alpha$-separated iff $A \cap \pi g\alpha\text{-}cl(B) = \phi$ and $\pi g\alpha\text{-}cl(A) \cap B = \phi$.

Lemma 4.4 : If E is a $\pi G\alpha$-connected subset of a topological space X such that $E \subset A \cup B$ where A and B are $\pi g\alpha$-separated sets then either $E \subset A$ or $E \subset B$.

Lemma 4.5 : In a topological space, if E is a $\pi G\alpha$-connected and F be any other set such that $E \subset F \subset \pi g\alpha\text{-}cl(E)$, then F is $\pi G\alpha$-connected.

Proof : Suppose F is not $\pi G\alpha$-connected. Then F can be written as the disjoint union of non-empty $\pi g\alpha$-closed sets G and H such that $F = G \cup H$. Since $E \subset F$ and E is $\pi G\alpha$-connected, $E \subset G$ or $E \subset H$. Let $E \subset G$. Then $\pi g\alpha\text{-cl}(E) \subset \pi g\alpha\text{-cl}(G)$

$\Rightarrow \pi g\alpha\text{-cl}(E) \cap H \subset \pi g\alpha\text{-cl}(G) \cap H \Rightarrow \pi g\alpha\text{-cl}(E) \cap H = \phi$. Also, $F \subset \pi g\alpha\text{-cl}(E)$

$\Rightarrow G \cup H \subset \pi g\alpha\text{-cl}(E) \Rightarrow H \subset \pi g\alpha\text{-cl}(E) \Rightarrow H \subset \phi \Rightarrow H = \phi$ which is a contradiction.

Hence F is $\pi g\alpha$-connected.

Theorem 4.6 : If $f : X \to Y$ is $\pi g\alpha$-connected, injective, M-$\pi g\alpha$-open map and $G(f)$ is $\pi g\alpha$-closed then X is $\pi g\alpha$-T_2 provided it is T_1 and locally $\pi G\alpha$-connected.

Proof : Let $x_1, x_2\ (\neq x_1) \in X$. Since f is injective, $f(x_1) \neq f(x_2)$.

Hence $(x_1, f(x_2)) \in X \times Y$-$G(f)$. Since $G(f)$ is $\pi g\alpha$-closed by lemma 3.2 there exist $U \in \pi G\alpha O(X, x_1)$ and $V \in \pi G\alpha O(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Since X is locally $\pi G\alpha$-connected there exist a $\pi G\alpha$-connected, set U_1 such that $x \in U_1 \subset U$. Then it follows that $f(U_1) \cap V = \phi$. Since f is M-$\pi g\alpha$-open, $f(U_1)$ is $\pi g\alpha$-open. **Claim :** $x_2 \notin \pi g\alpha$-cl(U_1) Suppose $x_2 \in \pi g\alpha$-cl (U_1). Since X is T_1, $\{x_2\}$ is a closed set and hence is $\pi g\alpha$-closed. Thus $U_1 \subset U_1 \cup \{x_2\} \subset \pi g\alpha$-cl $(U_1 \cup \{x_2\}) \subset \pi g\alpha$-cl (U_1). Hence by Lemma 4.5 $U_1 \cup \{x_2\}$ is $\pi G\alpha$-connected. Since f is $\pi g\alpha$-connected, $f(U_1 \cup \{x_2\}) = f(U_1) \cap f(\{x_2\})$ is $\pi G\alpha$-connected in Y which is absurd as $f(U_1)$ and V are $\pi g\alpha$-open sets such that $f(U_1) \cap V = \phi$ which is a contradiction. Hence $x_2 \notin \pi g\alpha$-cl(U_1). Setting $U_0 = X$-$\pi g\alpha$-cl(U_1) we find $x_2 \in U_0$. Thus $U_1 \in \pi G\alpha O\ (X, x_1)$ and $U_0 \in \pi G\alpha O\ (X, x_2)$ with $U_1 \cap U_0 = \phi$. Hence X is $\pi g\alpha$-T_2.

Theorem 4.7 : If for the function $f : X \to Y$ where Y is $\pi G\alpha O$ compact relative to Y, $G(f)$ is $\pi g\alpha$-closed in $X \times Y$ then f is $\pi g\alpha$-continuous.

Proof : Let $x \in X$. Let V be open in Y and $y \in Y$-V. Then $(x, y) \in X \times Y$-$G(f)$. Since $G(f)$ is $\pi g\alpha$-closed, there exist $U_y \in \pi G\alpha O(X, x)$ and $Vy \in \pi G\alpha O(Y, y)$, such that $f(U_y) \cap Vy = \phi$. This holds for every $y \in Y$-V. Clearly $\ell = \{Vy : y \in Y\text{-}V\}$ is a cover of Y-V by $\pi g\alpha$-open sets. Now Y is $\pi G\alpha O$-compact Y-V is $\pi G\alpha O$-closed. Hence by lemma 2.8 Y-V is $\pi G\alpha O$-compact relatively to Y. So ℓ has a finite sub family $\{V_{yi} : I = 1 \ldots n\}$ such that $Y\text{-}V \subset \bigcup_{i=1}^{n} V_{yi}$ Let $\{U_{yi} : i = 1 \ldots n\}$ be the corresponding sets of $\pi G\alpha O(X, x)$ satisfying $f(U_{yi}) \cap V_{yi} = \phi$. Set $U = \bigcap_{i=1}^{n} U_{yi}$. Now, $U \in \pi G\alpha O(X)$[1]. If $\alpha \in U$ then $f(\alpha) \in Vyi$ for all $i = 1 \ldots n$. This implies $f(\alpha) \notin \cup Vyi$. So that $f(\alpha) \notin Y$-V and hence $f(\alpha) \in V$. Since α is arbitrary, it follows that $f(U) \subset V$ which implies f is $\pi g\alpha$-continuous [5].

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