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ANALYTICAL SOLUTION OF CAUCHY'S SINGULAR INTEGRAL EQUATION FOR CLOSED AND UNCLOSED CONTOUR

By

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Abstract

In this paper, we have solved the Cauchy's singular integral equation for closed and unclosed contour with the help of the Poincare-Bertrand Transformation formula (PBTF) and the Plemelj formula (PF). Also we have generalized some sigular integral equations and their solutions applying different conditions on the original sigular integral equations. Applications of these sigular integral equations in Boundary value problems of Elasticity and allied subjects are well known.

Key words: Holder condition, PBT formular, Plemelj formula, Cauchy kernel

1. Introduction

Integral equations are one of the most useful techniques in many branches of pure and applied mathematics, particularly on account of its importance in boundary value problems in the ordinary and partial differential equations. Integral

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equations occur in many fields of mechanics and mathematical physics. They ar also related with the problems in mechanical vibrations, theory of analytic functions orthogonal systems, and quadratic forms of infinitely many variables. Integra equations arise in several problems of science and technology and may be obtaine directly from physical problems, i.e., radiation transfer problem and neutro diffusion problem etc. They also arise as representation formula for the solution of differential equations; a differential equation can be replaced by an integra equation with the hep of initial and boundary conditions. As such, each solution c the integral equation automatically satisfies these boundary conditions Jeffry [2 Swarup [8], Tricomi [9], Wirda [12].

The name 'Integral Equation' for any equation involving the unknow function $\varphi(x)$ under the integral sign was introduced by Bois-reymond in 1888. I 1782, Laplace used the integral transfor $f(x)=\int_0^\infty e^{-xt} f(t)dt$ to solve the linea integral equations and differential equations. In 1826, Abel solved the singul integral equation named after him having the from

$$f(x)=\int_0^x \frac{\varphi(t)}{(x-t)^\alpha}\,dt$$

Where $f(x)$ is a continuous function satisfying $f(a) = 0$ and $0 < \alpha < 1$. Huyge solved the Abel's integral equation for $\alpha = 1/2$. In 1826, Poission obtained integral equation of the type

$$\varphi(x)=f(x)+\lambda\int_0^x k(x,t)\varphi(t)\,dt$$

in which the unknown function $\varphi(t)$ occurs outside as well as before the integ sign and the variable x appears as one of the limits of the integral. Dirichle problem, which is the determination of a functions ψ having prescribed values over a certain boundary surface S and satisfying Laplaces equations $\nabla^2\psi = 0$ within the region enclosed by S, was shown by Heumann in 1870 to be equivalent to the solution of an integral equation. He solved the integral equation by an expansion in powers of a certain parameter λ. In 1896, Italian mathematician V. Volterra gave the first general treatment of the solution of the class of linear intergal equation bearing his name and characterized by the variable x appearing as the upper limit of the integral. In 1900, Swedish mathematician I. Fredholm have discussed a more general class of linear integral equation having the form

$$\varphi(x) = f(x) + \lambda\int_a^b k(x,t)\varphi(t)dt$$

The domain of the integral equation is growing up very quickly. A lot of mathematical papers and practical trials are published every month Mikhlin [3], Pogorzelski [7], Widom [10].

In this paper, we have included relatively two new technique namely, Poincare-Bertrand Transformation formula (PBTF) and Plemelj formula (PF) for determining the solution of Cauchy's singular integral equation for closed and unclosed contour restectively.

2. Integral Equation:

An integral equation is an equation in which an unknown function is to be determined appears under one or more integral signs.

Examples: 1. $\varphi(x) = F(x) + \lambda\int_a^b k(x,t)\varphi(t)dt$

2. $\varphi(x) = \lambda\int_a^b k(x,t)\varphi(t)dt$

The function $\varphi(x)$ in examples (1) and (2) as the unknown function to be determining, while all other functions are known functions. The function $k(x,t)$ is called the kernel of the integral equation and λ is non-zero real or complex paramete

If Ω is the domain of the variable t, we are often write an integral equation as,

$$\varphi(x) = F(x) + \lambda \int_{\Omega} k(x,t)\varphi(t)dt$$

3. Singular Integral Equation

An integral equation of the form $\varphi(x) = f(x) + \lambda \int_{C} k(x,t)\varphi(t)dt$ is said be singular if the range of integration is infinite i.e., defined in the integral $0 < x <$ or $-\infty < x < \infty$ or in which the kernel is discontinuous, i.e., the kernel is not squa integrable.

For Example

1. $f(x) = \int_0^\infty Sin(x-t)\varphi(t)dt$

2. $f(x) = \int_0^\infty e^{-xt}\varphi(t)dt$ and

3. $f(x) = \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}dt$

In example (1) and (2) the range of integration is infinite, while in example (3) range of integration is finite but the kernel is unbounded.

4. Cauchy Integrals:

The integral equation of the from $f(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(s)}{s-z}ds$ . . . (1)

where C is a regular curve is known as Cauchy type integral equation.

5. Holder Condition

A function $f(x)$ is said to satisfy the Holder condition if there exist constants M and μ, $0 < \mu \leq 1$ such that for every pair of points x_1 and x_2 lying in the range $a \leq x \leq b$, have $|f(x_1) - f(x_2)| < M |x_1 - x_2|^{\mu}$ Muskhelishvilli [5], Pennline [4].

6. Holder Continuous Function

A function which satisfies the Holder's condition is known as Holder continuous function.

7. Plemelj Formula (PF)

Let $\varphi(s)$ be a Holder continuous function of a point on a regular closed contour C and let a point z tend, in any arbitrary manner, from inside or outside the contour C, to the point t on this contour, then the integral (1) tends to the limit

$$f^{+}(t) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_C \frac{\varphi(s)}{s-t}\,ds \qquad \ldots (2)$$

or $$f^{-}(t) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_C \frac{\varphi(s)}{s-t}\,ds$$ Chakrabarti [1] ... (3)

respectively. The above formulas (2) and (3) are known as Plemelj formula. It follows that the boundary value $f^{+}(t)$ relates to the values of the Cauchy integral inside the region bounded by C, while the second boundary value $f^{-}(t)$ relates to the value in the outside region.

8. Poincare-Bertrand Transformation Formula (PBTF)

Let $\varphi(t)$ be Holder continuous function and C be a closed contour, then

$$\frac{1}{(2\pi i)^2}\int_C \frac{ds_1}{s_1 - t}\int_C \frac{\varphi(s)}{s - s_1}\,ds = \frac{1}{4}\varphi(t)$$ Williams [11], Peters [6]

... (4)

is known as Poincare-Bertrand Transformation Formula (PBTF).

9. Solution of Cauchy's Singular Integral Equation for Closed Contour

Consider the singular integral equation of second kind

$$ay(x) = f(x) - \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt \qquad \ldots (5)$$

where a and b are known complex constants, $y(t)$ is a Holder continuous functio and C is contour.

Equation (5) can be written as

$$ay(x) + \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt = f(x) \qquad \ldots (6)$$

Define an operator L such that

$$Ly = ay(x) + \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt \qquad \ldots (7)$$

Using (6) and (7), we have

$$Ly = f(x) \qquad \ldots (8)$$

Consider an adjoint operator M define by

$$M\varphi = a\varphi(x) - \frac{b}{\pi i}\int_C \frac{\varphi(\sigma)}{\sigma - x}\,d\sigma \qquad \ldots (9)$$

From (8), we have

$$M(Ly) = Mf$$

$$\Rightarrow M[ay(x) + \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt] = Mf \qquad \ldots (10)$$

Let $\varphi(x) = ay(x) + \dfrac{b}{\pi i}\displaystyle\int_C \dfrac{y(t)}{t-x}\,dt$ . . . (11)

$\Rightarrow \varphi(\sigma) = ay(\sigma) + \dfrac{b}{\pi i}\displaystyle\int_C \dfrac{y(t)}{t-a}\,dt$ . . . (12)

From (10) and (11), we have,

$$M\varphi = Mf$$

$$\Rightarrow a\varphi(x) - \frac{b}{\pi i}\int_C \frac{\varphi(\sigma)}{\sigma - x}\,d\sigma = af(x) - \frac{b}{\pi i}\int_C \frac{f(\sigma)}{\sigma - x}\,d\sigma$$

$$\Rightarrow a\{ay(x) + \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt\} -$$

$$\frac{b}{\pi i}[\int_C \frac{1}{\sigma - x}\{ay(\sigma) + \frac{b}{\pi i}\int_C \frac{y(t)}{t-\sigma}\,dt\}]d\sigma = af(x) - \frac{b}{\pi i}\int_C \frac{f(\sigma)}{\sigma - x}\,d\sigma$$

$$\Rightarrow a^2 y(x) + \frac{ab}{\pi i}\int_C \frac{y(t)}{t-x}\,dt - \frac{ab}{\pi i}\int_C \frac{y(\sigma)}{\sigma - x}\,d\sigma$$

$$-\frac{b^2}{(\pi i)^2}\int_C \frac{d\sigma}{\sigma - x}\int_C \frac{y(t)}{t-\sigma}\,dt = af(x) - \frac{b}{\pi i}\int_C \frac{f(\sigma)}{\sigma - x}\,d\sigma$$

Now using Poincare-Bertrand fomula, we have,

$$a^2 y(x) - 4b^2.\frac{1}{4}y(x) = af(x) - \frac{b}{\pi i}\int_C \frac{f(\sigma)}{\sigma - x}\,d\sigma$$

$$\Rightarrow (a^2 - b^2)y(x) = af(x) - \frac{b}{\pi i}\int_C \frac{f(t)}{t-x}\,dt$$

$\Rightarrow y(x) = \dfrac{a}{(a^2 - b^2)}f(x) - \dfrac{b}{(a^2 - b^2)\pi i}\displaystyle\int_C \dfrac{f(t)}{t-x}\,dt$ . . . (13)

which gives the solution of the integral equation.

10. Solution of Cauchy's Singular Integral Equation for Unclosed Contour:

Since the contour C is not closed, therefore Poincare-Bertrand method is not applicable. Here Plemelj formula is valid. Plemelj formulas (2) and (3) in article 7 still hold for an arc when we define the plus and minus directions as follows. To this end, we supplement the arc C with another arc C' so as to form a closed contour $C+C'$. Then, the interior and exterior of this closed contour stand for the plus and minus directions. Accordingly, for an arc C, we have

$$f^{+}(x)=\frac{1}{2}y(x)+\frac{1}{2\pi i}\int_C \frac{y(t)}{t-x}dt \quad \ldots (14)$$

and $$f^{-}(x)=-\frac{1}{2}y(x)+\frac{1}{2\pi i}\int_C \frac{y(t)}{t-x}dt \quad \ldots (15)$$

Subtracting (14) from (15), we got

$$y(x)=f^{+}(x)-f^{-}(x) \quad \ldots (16)$$

Again adding (14) and (15), we get

$$\frac{1}{\pi i}\int_C \frac{y(t)}{t-x}dt=f^{+}(x)+f^{-}(x) \quad \ldots (17)$$

Consider that a function $w(x)$ prescribed on an arc C and that it satisfy the Holde condition on C We shall find a function $w(z)$ analytic for all points z on C such th if satisfies the boundary or Jump condition

$$w^{+}(x)-w^{-}(x)=w(x);\forall x\in C \quad \ldots (18)$$

The problem (18) is a special case of (17) so called Riemann-Hilbert proble which requires the determination of a function $w(z)$ analytic for all points z n lying on C such that for x on C.

$$w^{+}(x)-z(x)w^{-}(x)=w(x) \quad \ldots (19)$$

where $w(x)$ and $z(x)$ are given complex valued function. The integral equation is given as

$$ay(x) = F(x) - \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt \qquad \ldots (20)$$

Using (16), (17) in (20), we get

$$(a+b)f^{+}(x) - (a-b)f^{-}(x) = F(x) \qquad \ldots (21)$$

Consider C be a regular unclosed contour then the solution of the singular integral equation (14) is given as

$$y(x) = \frac{a}{(a^2-b^2)} f(x) + \frac{c}{(x-\alpha)^{1-m}(x-\beta)^m} - \frac{b}{(a^2-b^2)\pi i}\left(\frac{x-\alpha}{x-\beta}\right)^m \int_C \left(\frac{t-\beta}{t-\alpha}\right)^m f(x).\frac{dt}{t-x} \qquad \ldots (22)$$

where α and β are the begining and end points of the contour C and the number m is given by

$$m = \frac{1}{2\pi i}\log\left(\frac{a+b}{a-b}\right)$$

11. Results and Discussion

In this section, we have considered some interesting generalizations applying different conditions on the equation (5) for closed contour and the equation (22) for unclosed contour.

Generalization 1:

The Cauchy's singular integral of second kind given by (5) as

$$ay(x) = f(x) - \frac{b}{\pi i}\int_C \frac{y(t)}{t-x}\,dt \qquad \ldots (E1)$$

an its solution is given by (13) as

$$y(x) = \frac{a}{(a^2 - b^2)} f(x) - \frac{b}{(a^2 - b^2)\pi i} \int_C \frac{f(t)}{t - x} dt \qquad \ldots (E2)$$

Case 1:

If we consider $a = 0$ in (E1), then equation (E1) reduces to the first kind as

$$f(x) = \frac{b}{\pi i} \int_C \frac{y(t)}{t - x} dt \qquad \ldots (E3)$$

and its solution is given by

$$y(x) = \frac{1}{b\pi i} \int_C \frac{f(t)}{t - x} dt \qquad \ldots (E4)$$

Case 2:

If we consider $a = 0$ and $b = 1$ in (E1), then equation (E1) becomes

$$f(x) = \frac{1}{\pi i} \int_C \frac{y(t)}{t - x} dt \qquad \ldots (E5)$$

and its solution is given by

$$y(x) = \frac{1}{\pi i} \int_C \frac{f(t)}{t - x} dt \qquad \ldots (E6)$$

Generalization 2:

The Cauchy's singular integral of second kind for unclosed contour studied by (14) as

$$f(x) = \frac{1}{2} y(x) + \frac{1}{2\pi i} \int_C \frac{y(t)}{t - x} dt \qquad \ldots (E7)$$

and its solution is given by (22) as

$$y(x)=\frac{a}{(a^2-b^2)}f(x)+\frac{C}{(x-\alpha)^{1-m}(x-\beta)^m}-\frac{b}{(a^2-b^2)\pi i}\left(\frac{x-\alpha}{x-\beta}\right)^m\int_C\left(\frac{t-\beta}{t-\alpha}\right)^m f(x).\frac{dt}{t-x} \quad \text{. . . (E8)}$$

Case 1:

If we putting b =1 in (E8), then solution (E8) reduces to the singular integral equation of first kind as

$$y(x)=\frac{a}{(a^2-1)}f(x)+\frac{C}{(x-\alpha)^{1-m}(x-\beta)^m}-\frac{b}{(a^2-1)\pi i}\left(\frac{x-\alpha}{x-\beta}\right)^m\int_C\left(\frac{t-\beta}{t-\alpha}\right)^m f(x).\frac{dt}{t-x} \quad \text{. . . (E9)}$$

Case 2:

If $a = 0$, $b = 1$, and $m=\frac{1}{2}$ in (E8), then the solution (E8) reduces to

$$y(x)=\frac{1}{\pi i}\left(\frac{x-\alpha}{x-\beta}\right)^{\frac{1}{2}}\int_C\left(\frac{t-\beta}{t-\alpha}\right)^{\frac{1}{2}}.\frac{f(t)}{t-x}dt+\frac{A}{[(x-\alpha)(x-\beta)]^{\frac{1}{2}}} \quad \text{. . . (E10)}$$

where A is any arbitrary constant.

12. Conclusion

From our above discussion we have seen that both Poincare-Bertrand Transformation formula (PBTF) and Plemelj formula (PF) is applicable for the singular integral equation whose kernel is of the form $k(x,t)=\frac{1}{t-x}$. Also we conclude that Poincare-Bertrand Transformation formula (PBTF) is valid for solving the Cauchy's singular integral for the closed contour, But (PBTF) is not valid for unclosed contour. On the other hand, Plemelj formula (PF) is valid for unclosed contour, Also Plemelj formula (PF) is valid for closed contour after constructing the closed contour connecting by more than one unclosed contour.

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