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A DECOMPOSITION OF A WEAKER FORM OF CONTINUITY

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Abstract

The aim of this paper is to give a decomposition of a weaker form of continuity namely πg-continuity and $\pi g\alpha$-continuity by introducing the concepts of C_π and K_π-sets.

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1. Introduction

The decomposition of continuity is one of the many problems in general topology. Tong [13] introduced the notions of A-sets and A-continuity and established a decomposition of continuity. Also Tong [14] introduced the notion of

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B sets and B-continuity and used them to obtain another decomposition of continuity. Ganster and Reilly [6] have improved Tong's decomposition result. Recently, Dontchev, Przemski [4] and Rajamani [12] obtained some more decomposition of continuity.

In this paper, we introduced the notions of C_π - sets, C_π^* - sets and K_π - sets, K_π^*-sets to obtain decompositions of πg-continuity, contra-πg-continuity, πg-open maps and $\pi g\alpha$-continuity, $\pi g\alpha$-open maps respectively.

2. Preliminaries

Throughout this paper, X and Y denote topological spaces (X,τ) and (Y,σ) respectively on which no separation axioms are assumed. If A is a subset of a topological space (X,τ) int A and cl(A) denote the interior and closure of A in a topological space X. A subset A is said to be regular open if A = intcl(A). The finite union of regular open sets is said to be π-open.

First we recall some definitions which are used in this paper.

Definition 2.1: A subset A of a topological space X is said to be

(a) πg-closed [5] if cl$(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

(b) $\pi g\alpha$-closed [1] if αcl$(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

(c) $\pi g\alpha$-closed [11] if pcl$(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

(d) rg-closed [9] if cl$(A) \subset U$ whenever $A \subset U$ and U is regular-open.

(e) g-closed [7] if cl$(A) \subset U$ whenever $A \subset U$ and U is open.

(f) A-set [13] if $A = G \cap F$ where G is open and F is regular-closed in X.

(g) B-set [14] if $A = G \cap F$ where G is open and F is a t-set.

(h) t-set [14] if int A = intcl(A)

(i) α^*-set [8] if int A = int(cl(int A))

(j) LC-set [3] if $A = G \cap F$ where G is open and F is a closed set in X.

(k) C-set [12] if $A = G \cap F$ where G is g-open and F is a t-set in X.

(l) C_r-set [12] if $A = G \cap F$ where G is rg-open and F is a t-set.

(m) C_r*-set [12] if $A = G \cap F$ where G is rg-open and F is a α^*-set in X.

Definition 2.2: A subset A of a topological space X is said to be πg-open [5] (resp. $\pi g\alpha$-open, πgp-open, g-open, rg-open) if its complement X-A is πg-closed (resp. $\pi g\alpha$-closed, πgp-closed, g-closed, rg-closed).

Definition 2.3 : A function $f: X \to Y$ is said to be $\pi g\alpha$-continuous [2] (resp. πg-continuous [5], πgp-continuous [10]) if $f^{-1}(V)$ is $\pi g\alpha$-open in X (resp. πg-open, πgp-open) for every open set V of Y.

Definition 2.4 : A map $f: X \to Y$ is said to be $\pi g\alpha$-open-map (resp. πgp-open map, πg-open map) if $f(U)$ is $\pi g\alpha$-open in Y resp. πgp-open, πg-open) for every open set U is X.

Lemma 2.5: [5] A subset A of a space X is πg-open iff $F \subset \text{int}A$ whenever F is π-closed and $F \subset A$.

Lemma 2.6: [1] A subset A of a space X is $\pi g\alpha$-open iff $F \subset \alpha\, \text{int}A$ whenever F is π-closed and $F \subset A$.

Lemma 2.7: [11] A subset A of a space X is πgp-open off $F \subset \text{pint}A$ whenever F is π-closed and $F \subset A$.

Lemma 2.8: [12] Let A and B be subsets of a space X. If B is α^*-set then $\alpha\ \text{int}(A \cap B) = \alpha\ \text{int}\, A \cap \text{int}B$.

3. Weaker Sets

Definition 3.1: A subset S of (X,τ) is called a

1. C_π-set if $S = G \cap F$ where G is πg-open and F is a t-set.
2. C_π^*-set if $S = G \cap F$ where G is πg-open and F is a α^*-set.

3. K_π-set if $S = G \cap F$ where G is $\pi g\alpha$-open and F is a t-set.
4. K_π^*-set if $S = G \cap F$ where G is $\pi g\alpha$-open and F is a α^*-set.

Proposition 3.2:

1. Every B set is C_π-set
2. Every C set is C_π-set
3. Every C set is C_π^*-set
4. Every C_π -set is C_π^*-set
5. Every C_π -set is K_π-set
6. Every C_π -set is K_π^*-set
7. Every C_π -set is C_r-set
8. Every C_π^*-set is K_π^*-set
9. Every K_π -set is K_π^*-set

Proof : Straight forward

Converse of the above need not be true as seen in the following examples.

Example 3.3: Let $X = \{a,b,c\}, \tau = \{\phi, X, \{a,b\}\}$

$A = \{a,c\}$ is a C_π-set, C_π^*-set but neither a B set nor a C-set.

Example 3.4: Let $X = \{a,b,c,d\}, \tau = \{\phi, X\{a\}, \{b,c\}, \{a,b,c\}\}$

$A = \{c,d\}$ is a C_π^*-set, K_π^*-set. C_r-set, C_r^*-set but neither C_π-set nor K_π-set.

Example 3.5: Let $X = \{a,b,c,d\}, \tau = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{d\}, \{a,d\}\}$

$A = \{a,b,d\}$ is K_π-set, K_π^*-set but not C_π-set, C_π^*-set, C_r^*-set, C-set

Remarks 3.6: K_π and C_r^* are Independent concepts follows from example 3.4 and 3.5.

Remark 3.7: K_π and C_π^* are Independent concepts follows from example 3.4 and 3.5.

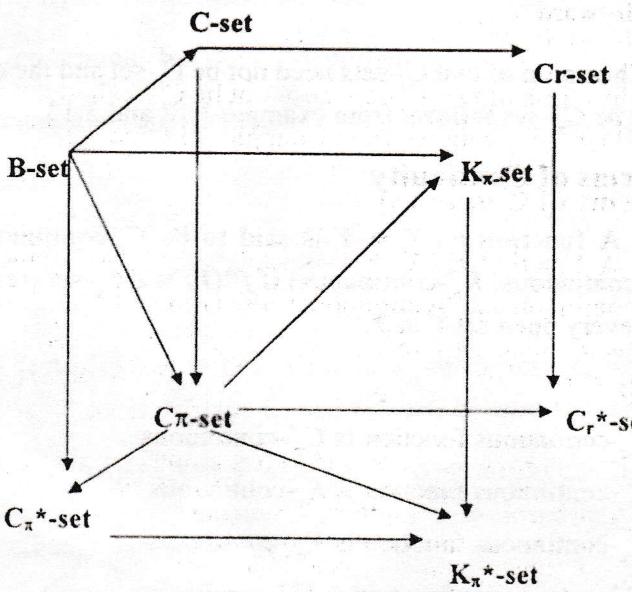
Proposition 3.8: If S is a $\pi g\alpha$-open set, then S is a K_π-set, K_π^*-set.

Proof: Obvious

However the converse of the above need not be true as seen in the following example.

Example 3.9: Let $X = \{a,b,c\}$, $\tau = \{\phi, X\{a\}, \{b\}, \{a,b\}\}$. Then $A = \{c\}$ is K_π, K_π^*-set but not $\pi g\alpha$-open.

The above discussion are summarised by the following diagram



Example 3.10: Let A and B be K_π-sets in X. Then $A \cap B$ is a K_π-set in X.

Proof: Since A is K_π-set, $A = G_1 \cap F_1$ where G_1 is $\pi g\alpha$-open and F_1 is t-set. B is a K_π-set $\Rightarrow B = G_2 \cap F_2$ where G_2 is $\pi g\alpha$-open and F_2 is t-set. Since intersection of two $\pi g\alpha$-open sets is $\pi g\alpha$-open and intersection of t-sets is t-set it follows that $A \cap B$ is a K_π-set in X.

Remark 3.11: 1. The Union of two K_π-sets need not be a K_π-set in X.

2. The Complement of a K_π-set need not be K_π-set.

Example 3.12: In example 3.4, $A = \{a,c\}$ and $B = \{d\}$ are K_π-sets. $A \cup B = \{a,c,d\}$ is not K_π-set.

Example 3.13: In example 3.4, X-$\{a,c\} = \{b,d\}$ is not K_π-set

Proposition 3.14: Let A and B be C_π-sets is X. Then $A \cap B$ is C_π-sets is X

Proof: Straight forward

Remark 3.15: The union of two C_π-sets need not be C_π-set and the complement of C_π-sets need not be C_π -set follows from example 3.12 and 3.13.

4. Weaker Forms of Continuity

Definition 4.1: A function $f : X \rightarrow Y$ is said to be C_π-continuous (resp. C_π^*-continuous, K_π-continuous, K_π^*-continuous) if $f^{-1}(V)$ is a C_π-set (resp. C_π^*-set, K_π-set, K_π^*-set) for every open set V in Y.

Proposition 4.2:

1. Every C_π-continuous function is C_π^*-continuous
2. Every C_π-continuous function is K_π-continuous
3. Every K_π-continuous function is K_π^*-continuous
4. Every C_π^*-continuous function is K_π^*-continuous

Proof: Obvious

The converses of above need not be true as can be seen from the following examples.

Example 4.3: Let $X = \{a,b,c,d\}, \tau = \{\phi, X, \{a\}, \{b,c\}, \{a,b,c\}\}, \sigma = \{\phi, X, \{c,d\}\}$ and $f: (X, \tau) \to (X, \sigma)$ be the identity mapping.

Then f is C_π^*-continuous but not C_π-continuous.

Example 4.4: Let $X=\{a,b,c,d\}$, $\tau=\{\phi, X, \{a\}, \{d\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$. $\sigma = \{\phi, X, \{a,b,d\}\}$ and $f: (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is K_π-continuous but not C_π-continuous.

Example 4.5: Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{b,c\}, \{a,b,c\}\}, \sigma = \{\phi, X, \{c,d\}, \{c\}\}$ and $f: (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is K_π^*-continuous but not K_π-continuous.

Example 4.6: Let $X=\{a,b,c,d\}, \tau = \{\phi, X, \{a\}, \{d\}, \{a,d\}\{c,d\}, \{a,c,d\}\}$ $\sigma=\{\phi, X, \{a,b,d\}, \{a\}\}$ and $f: (X, \tau) \to (X, \sigma)$ be the identity mapping.

Then f is K_π^*-continuous but not C_π^*-continuous.

The above discussions are summarized by the following implications:

$$\begin{array}{ccc} C_\pi\text{-continuity} & \Rightarrow & C_\pi^*\text{-continuity} \\ \Downarrow & & \Downarrow \\ K_\pi\text{-continuity} & \Rightarrow & K_\pi^*\text{-continuity} \end{array}$$

Definition 4.7: A map $f: X \to Y$ is said to be K_π-open (resp. C_π-open, C_π^*-open, K_π^*-open) if $f(U)$ is K_π-set in Y (resp. C_π-set, C_π^*-set, K_π^*-set) for each open set U in X.

Definition 4.8: A map $f: X \to Y$ is said to be contra-$\pi g\alpha$ continuous (contra-πgp-continuous, contra-πg-continuous) if $f^{-1}(V)$ is $\pi g\alpha$-closed in X(resp. πgp-closed, πg-closed) for each open set V in Y.

Definition 4.9: A map $f: X \to Y$ is said to be contra-K_π-continuous. (resp. contra-C_π-continuous, contra-C_π^*-continuous, contra-K_π^*-continuous) if $f^{-1}(V)$ is K_π-set (resp. C_π-set, C_π^*-set, K_π^*-set) for every closed set V in Y.

5. Decomposition of πg-continuity

Theorem 5.1: A subset S of X is

1. πg-open iff it is both πgp-open and C_π-set in X.
2. πg-open iff it is both $\pi g\alpha$-open and C_π-set in X.
3. πg-open iff it is both $\pi g\alpha$-open and C_π^*-set in X.

1. Proof: Necessity is trival.

Sufficiency: Assume that S is both πgp-open and C_π-set in X.

By assumption, S is a C_π-set in $X \Rightarrow S = A \cap B$ where A is πg-open and B is a t-set.

S is πgp-open and $F \subset S \Rightarrow F \subset \text{pint } S \subset \text{int}B$

A is πg-open and $F \subset S \subset A \Rightarrow F \text{ int}A$ [5]

Hence $F \subset \text{int } A \cap \text{int}B = \text{int}(A \cap B) = \text{int}S$.

Hence S is πg-open

2. Proof: Necessity is trivial

Sufficiency: Let S be both $\pi g\alpha$-open and C_π-set in X.

Since S is a C_π-set, $S = A \cap B$ where A is πg-open and B is a t-set.

Since S is $\pi g\alpha$-open, $F \subset S \Rightarrow F \subset \alpha \text{ int } S \subset \text{int}B$

A is πg-open and $F \subset S \subset A \Rightarrow F \subset \text{int}A$. Hence

$F \subset \text{int}A \cap \text{int}B = \text{int}(A \cap B) = \text{int}S$.

3. Proof: Necessity is trivial

Sufficiency: Assume S is both $\pi g\alpha$-open and C_π^*-set in X.

Since S is C_π^*-set in X, $S = A \cap B$ where A is πg-open and B is α^*-set in X.

Since S is $\pi g\alpha$-open, $F \subset S \Rightarrow F \subset \alpha \text{ int } S \subset \text{int}B$

A is πg-open and $F \subset S \subset A \Rightarrow F \subset \text{int } A$. Hence $F \subset \text{int } A \cap \text{int}B = \text{int}S$.

Theorem 5.2: A mapping $f : X \to Y$ is

1. πg-continuous iff it is both πgp-continuous and C_π-continuous
2. πg-continuous iff it is both $\pi g\alpha$-continuous and C_π-continuous
3. πg-continuous iff it is both $\pi g\alpha$-continuous and C_π^*-continuous

Proof: follows from theorem 5.1

Theorem 5.3: A map $f : X \to Y$ is

1. πg-open iff it is both πgp-open and C_π-open
2. πg-open iff it is both $\pi g\alpha$-open and C_π-open
3. πg-open iff it is both $\pi g\alpha$-open and C_π^*-open

Proof: follows from theorem 5.1

Theorem 5.4: A mapping $f : X \to Y$ is

1. Contra-πg-continuous iff f is both contra-πgp-continuous and contra-C_π-continuous
2. Contra-πg-continuous iff f is both contra-$\pi g\alpha$-continuous and contra-C_π-continuous
3. Contra-πg-continuous iff f is both contra-$\pi g\alpha$-continuous and contra-C_π^*-continuous

Proof: follows from theorem 5.1

6. Decomposition of $\pi g\alpha$-continuity

In this section we have obtained two decomposition of $\pi g\alpha$-continuity, $\pi g\alpha$-open maps and contra-$\pi g\alpha$-continuity.

Theorem 6.1: A subset S of X is

(a) $\pi g\alpha$-open iff it is both πgp-open and a K_π-set

(b) $\pi g\alpha$-open iff it is both πgp-open and a K_π^*-set

Proof: Necessity : Let S be $\pi g\alpha$-open.

For any subset A of X, int $A \subset \alpha$ int $A \subset p$ int$A$.

Since S is $\pi g\alpha$-open, $F \subset S \Rightarrow F \subset \alpha$ int $S \subset p$int $S \Rightarrow S$ is πgp-open.

Since $S = S \cap X$ where S is $\pi g\alpha$-open and X is a t-set, S is a K_π-set.

Sufficiency: Let S be both πgp-open and a K_π-set.

Since S is a K_π-set, $S = A \cap B$ where A is $\pi g\alpha$-open and B is a t-set Since S is πgp-open, $F \subset S \Rightarrow F \subset p$ int $S = S \cap$ int cl(S) $\subset$ int$B$. A is $\pi g\alpha$-open and $F \subset S \subset A \Rightarrow F \subset \alpha$ int A. Therefore $F \subset \alpha$ int $A \cap$ int $B \subset \alpha$ int $(A \cap B) \subset \alpha$ int S.

(b) Proof: Similar as (a)

Theorem 6.2 : A map $f: X \to Y$ is

(a) $\pi g\alpha$-continuous iff it is πgp-continuous and K_π-continuous

(b) $\pi g\alpha$-continuous iff it is πgp-continuous and K_π^*-continuous

Proof: Follows from theorem 6.1

Theorem 6.3: A map $f: X \to Y$ is

(a) $\pi g\alpha$-open iff it is πgp-open and K_π-open

(b) $\pi g\alpha$-open iff it is πgp-open and K_π^*-open

Proof: Follows from theorem 6.1

Theorem 6.4: A map $f: X \to Y$ is

(a) Contra-$\pi g\alpha$-continuous iff it is both contra-πgp-continuous and contra-K_π-continuous

(b) Contra-$\pi g\alpha$-continuous iff it is both contra-πgp-continuous and contra-K_π^*-continuous

Proof: Follows from theorem 6.1

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