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FLUID DYNAMICS OF A CIRCULAR POROUS SLIDER FOR NON-NEWTONIAN FLUIDS AND FOR NEWTONIAN FLUIDS UNDER A TRANSVERSE MAGNETIC FIELD

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ABSTRACT

The three-dimensional flow of fluids of constant density forced through the porous bottom of a circular slider, which is moving laterally on a flat plate, has been discussed for two cases (*i*) when the fluid is non-Newtonian (*ii*) when the fluid is Newtonian and electrically conducting under the effect of a transverse magnetic field. Assuming the cross-flow Reynolds number to be small, all entities have been expanded in its powers. It is found that with the increase of non-Newtonian parameter the lift on the slider increases while the drag on it decreases. Also in hydromagnetic case, the lift on the slider increases and the drag on it decreases with the increase of Hartmann number. This shows that the effects of a transverse magnetic field on an electrically conducting Newtonian fluid are qualitatively similar to those of the non-linear terms in the constitutive equations of the fluid.

1. *Introduction*

It has been observed in a number of flow problems that the effects of a transverse magnetic field on the flow pattern and stresses in the flow of a Newtonian fluid are qualitatively similar to those of including second-order (non-Newtonian) or elastico-viscous terms in the constitutive equation of the fluid.

The aim of the present paper is to compare the effects produced by a transverse magnetic field in the flow of an electrically conducting Newtonian fluid and those produced by non-linear terms in the constitutive equation of a second-order fluid [1], when such fluids are forced through the porous bottom of a circular slider which is moving on a flat plate with a uniform velocity. This problem has been solved in Newtonian fluid by Chang Yi Wang [2]. We have solved the equations by expanding the velocity components in the powers of the cross-flow Reynolds number. It has been found that the application of a transverse magnetic field increases the load bearing capacity of the system and similar is the effect of the non-Newtonian terms.

2. *Formulation of the problem*

We consider the flow of a fluid confined between a circular porous slider and a flat plate. The fluid is forced through the porous bottom of a circular slider of radius 'a' which is moving laterally with a constant velosity U above the fixed flat plate. It is assumed that the gap width 'd' between the circular slider and the flat plate is small compared with the radius of the slider such that the edge effects are negligible.

We choose cartesian co-ordinates (x,y,z) translating with the slider. Hence the slider becomes fixed at $z=d$ injecting a fluid with velocity W. The plate is the plane $z=0$ moving in the direction of x with the velocity U. Taking u, v, w to be the velocity components of the fluid in the directions of x, y, z respectively, the boundary conditions of the problem are

$$\left.\begin{array}{llll} u=U, & v=0, & w=0, & \text{at } z=0 \\ u=0, & v=0, & w=-W, & \text{at } z=d \end{array}\right\} \tag{1}$$

The above boundary conditions suggest the following form of the velocity components

$$u=Uf(\eta)+\frac{W}{d}\,xh'(\eta) \tag{2}$$

$$v=\frac{W}{d}\,yh'(\eta) \tag{3}$$

$$w=-2Wh(\eta) \tag{4}$$

where $\eta=\frac{z}{d}$ and a prime denotes differentiations with respect to η. The boundary conditions in terms of f and h are

$$\left.\begin{array}{llll} f=1, & h=0, & h'=0 & \text{at } \eta=0 \\ f=0, & h=\frac{1}{2}, & h'=0 & \text{at } \eta=1 \end{array}\right\} \tag{5}$$

3. *Flow of a second-order fluid*

The constitutive equation of an incompressible second-order fluid has been suggested by Coleman and Noll [1] as

$$\tau_{ij}=-p\delta_{ij}+\mu_1\overset{(1)}{e_{ij}}+\mu_2\overset{(2)}{e_{ij}}+4\mu_3\overset{(1)}{e_{i\alpha}}\overset{(1)}{e_{\alpha j}}\,g^{\alpha\alpha} \tag{6}$$

where

$$\overset{(1)}{e_{ij}}=\tfrac{1}{2}(v_{i,j}+v_{j,i}) \tag{7}$$

$$\overset{(2)}{e_{ij}}=\tfrac{1}{2}(a_{i,j}+a_{j,i}+2v_{m,i}v_{m,j}g^{mm}) \tag{8}$$

τ_{ij} is the stress tensor, v_i, a_i are velocity and acceleration vectors respectively; μ_1, μ_2, μ_3 are the material constants and p is an undetermined hydrostatic pressure. The tensors $\overset{(1)}{e_{ij}}$ and $\overset{(2)}{e_{ij}}$ are known as the first and second rate of strain tensors. Makovitz and Coleman [3] have shown from thermodynamic considerations that μ_2 should be negative. It is found experimentally that the numerical value of μ_2 is always less than μ_3 (see Truesdell [4]). To simplify the calculations we assume here that

$$\mu_2 = -\tfrac{1}{2}\mu_3 \tag{9}$$

The equation of continuity and Cauchy's equation of motion are given by

$$v^i{}_{,j} = 0 \tag{10}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v^j{}_i v_{,j}\right) = F_i + \tau^j{}_{i,j} \tag{11}$$

where ρ is the density of the fluid and F_i is the external force vector.

The equation of continuity is automatically satisfied by the form of the velocity components given by (1), (2), and (3). Taking the external force vector F_i to be zero and substituting (2)—(4) and (6)—(8) in the equation (11), we get the following set of equations in the directions of x, y and z respectively

$$\rho\frac{UW}{d}\left[(fh' - 2hf') - \frac{f''}{R} + \alpha(3h'f'' + h'''f - 2hf''')\right]$$

$$= -\frac{\partial p}{\partial x} + \frac{\rho x W^2}{d^2}\left[\frac{h'''}{R} - (h'^2 - 2hh'') + \alpha(2hh'''' - 4h'h''')\right] \tag{12}$$

$$\rho\frac{yW^2}{d^2}\left[(h'^2 - 2hh'') - \frac{h'''}{R} + \alpha(4h'h''' - 2hh'''')\right] = -\frac{\partial p}{\partial y} \tag{13}$$

$$\rho W^2\left[4hh' + \frac{2h''}{R} + \alpha(12h'h'' - 4hh''')\right] = -\frac{\partial p}{\partial \eta} \tag{14}$$

where $\nu_1 = \frac{\mu_1}{\rho}$, $\nu_3 = \frac{\mu_3}{\rho}$, $\alpha = \frac{\nu_3}{2d^2}$ and $R = \frac{Wd}{\nu_1}$. The equations (12)−(14) suggest the following form for the pressure

$$-\frac{p}{\rho} = W^2\left[2h^2 + \frac{2h'}{R} + \frac{K}{2d^2}(x^2 + y^2) + \alpha(8h'^2 - 4hh'')\right] + A \tag{15}$$

where K and A are absolute constants. With this form of the pressure, the isobars are concentric circles about z-axis which is consistent with the fact that the slider is circular. Putting the expression for the pressure from (15), the equations (12) and (13) give

$$f'' = R\left[(h'f - 2hf') + \alpha(fh''' + 3h'f'' - 3hf''')\right] \tag{16}$$

$$h''' = R\left[h'^2 - 2hh'') + \alpha(4h'h''' - 2hh'''') - K\right] \tag{17}$$

Differentiating (17) with respect to η we get

$$h'''' = R\left[-2hh''' + \alpha(4h''h''' + 2h'h'''' - 2hh''''')\right] \tag{18}$$

Assuming the Reynolds number R to be small, we expand $f(\eta)$ and $h(\eta)$ in its powers as

$$\left.\begin{aligned} f(\eta) &= \sum_{n=0}^{\infty} f_n(\eta) R^n \\ h(\eta) &= \sum_{n=0}^{\infty} h_n(\eta) R^n \end{aligned}\right\} \tag{19}$$

Boundary conditions (5) can be written as

$$\left.\begin{aligned} & f_o(0)=1,\ f_n(0)=0 \text{ for } n\geqslant 1, \\ & h_n(0)=0=h'_n(0) \text{ for all } n\,; \\ & h_o(1)=\tfrac{1}{2},\ h_n(1)=0 \text{ for } n\geqslant 1, \\ & f_n(1)=0=h'_n(1) \text{ for all } n. \end{aligned}\right\} \tag{20}$$

Substituting (19) and equating the coefficient of like powers of R in the equations (16) and (18), we get a set of linear equations in f_n and h_n. We have calculated their values upto $n=2$ which are given below :

$$h_0(\eta)=\frac{\eta^2}{2}(3-2\eta), \tag{21}$$

$$h_1(\eta)=\frac{1}{140}(13\eta^2-18\eta^3+7\eta^6-2\eta^7)+\frac{\alpha}{5}(12\eta^3-3\eta^2-15\eta^4+6\eta^5), \tag{22}$$

$$\begin{aligned} h_2(\eta)=&(0{\cdot}0019-0{\cdot}4071\alpha-2{\cdot}4343\alpha^2)\eta^2-(0{\cdot}0053-0{\cdot}7676\alpha-2{\cdot}0229\alpha^2)\eta^3 \\ &+\frac{1}{323400}(3080-1188\eta^7-1925\eta^9+1386\eta^{10}-252\eta^{11}) \\ &-\frac{\alpha}{2100}(1200\eta^4-648\eta^5+294\eta^6-684\eta^7+765\eta^8-170\eta^9) \\ &+\frac{\alpha^2}{175}(1470\eta^4-2898\eta^5+2100\eta^6-600\eta^7), \end{aligned} \tag{23}$$

$$f_0(\eta)=1-\eta, \tag{24}$$

$$f_1(\eta)=-\frac{1}{20}(6\eta-10\eta^3+5\eta^4-\eta^5)+\alpha(2\eta-3\eta^2+\eta^3), \tag{25}$$

$$\begin{aligned} f_2(\eta)=&[0{\cdot}1658-0{\cdot}5443\alpha+3{\cdot}2\alpha^2]\eta \\ &+\frac{1}{140}[4{\cdot}3333\eta^3-4{\cdot}5\eta^4+3\eta^5-35\eta^6+13{\cdot}5\eta^7-5{\cdot}375\eta^8+0{\cdot}82\eta^9] \\ &-\frac{\alpha}{140}[54\eta^2-32\eta^3-294\eta^4+240{\cdot}8\eta^5-49\eta^6+3\eta^7] \\ &+\frac{\alpha^2}{5}[36\eta^2-127\eta^3+105\eta^4-30\eta^5]. \end{aligned} \tag{26}$$

4. *A transverse magnetic field*

In this section, we consider that the fluid is an electrically conducting and stress is a linear function of strain rate tensor i.e., $\mu_2=\mu_3=0$ in the relation (6). We apply a constant magnetic field $H_0\vec{k}$ where $\vec{k}$ is the unit perpendicular to the plane of the slider. Assuming that the applied electric field is zero and the induced magnetic field is negligible, we get the external force vector $\vec{F}$ and the current density vector $\vec{J}$ as

$$\vec{F}=\mu_m \vec{J}\times(H_0 \vec{k}) \tag{27}$$

$$\vec{J}=\sigma\mu_m \vec{V}\times(H_0 \vec{k}) \tag{28}$$

where $H_0 \vec{k}$ is the magnetic field, $\vec{V}$ is the vector, σ is the electrical conductivity of the fluid and μ_m is the magnetic permeability. Substituting (2), (3), (4), (6) and (27), the equation (11) yields the following equations in the directions of x, y, z respectively

$$\frac{UW}{d}\left[(fh'-2hf')-\frac{f''}{R}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{xW^2}{d^2}\left[\frac{h'''}{R}-(h'^2-2hh'')\right]$$
$$+\frac{\mu_m}{\rho}J_yH_0 \tag{29}$$

$$y\frac{W^2}{d}\left[(h'^2-2hh'')-\frac{h'''}{R}\right]=-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\mu_m J_x H_0}{\rho} \tag{30}$$

$$\rho W^2\left[4hh'+\frac{2h''}{R}\right]=\frac{\partial p}{\partial \eta} \tag{31}$$

where J_x and J_y are components of $\vec{J}$ in the directions of x and y respectively. The equation (28) gives

$$\left.\begin{aligned} J_x&=\frac{\sigma\mu_m WH_0}{d}hy' \\ J_y&=-\sigma\mu_m H_0(Uf+\frac{W}{d}xh') \end{aligned}\right\} \tag{32}$$

while $J_z=0$. The equations (29)—(32) suggests the following expression for the pressure.

$$-\frac{p}{\rho}=W^2\left[2h^2+\frac{2h'}{R}+\frac{K}{2d^2}(x^2+y^2)\right]+A \tag{33}$$

which is the same as (15) with $\nu_3=0$. Substituting (32) and (33) in (29) and (30), we get

$$(f''-M^2f)=R(fh'-2hf') \tag{34}$$

$$(h'''-M^2h')=R(h'^2-2hh''-K) \tag{35}$$

where M is the Hartmann-number given by

$$M^2=\frac{\sigma\mu_m{}^2H_0{}^2d^2}{\rho\nu} \tag{36}$$

Assuming the cross-flow Reynolds number to be small and proceeding as in the section 3 we have calculated the expressions for f_0, f_1 and h_0, h_1' They are given as :

$$h_0(\eta)=\frac{\cosh M\eta-1+\cosh M-\cosh M(1-\eta)-M\eta\sinh M}{2[2(\cosh M-1)-M\sinh M]} \tag{37}$$

$$f_0(\eta)=\frac{\sinh M(1-\eta)}{\sinh M}, \tag{38}$$

$$\begin{aligned}h_1(\eta)=&-\frac{49a_1a_2}{12M}+2(\cosh M\eta-1)\\&\left[a_1A+a_2-\frac{MB}{2[2(\cosh M-1)-M\sinh M]}\right]\\&-\frac{(12a_1{}^2+19a_2{}^2)}{12}\eta+2(a_2A+a_1B)(\sinh M\eta-M\eta)\\&-\frac{1}{12M}[(a_1{}^2+a_2{}^2)\sinh 2M\eta+2a_1a_2\cosh 2M\eta\\&-12Ma_1\eta(a_1\cosh M\eta+a_2\sinh M\eta)-6M^2a_2\eta^2(a_1\cosh M\eta\\&+a_2\sinh M\eta)+30Ma_2\eta(a_1\sinh M\eta+a_2\cosh M\eta)\\&-51a_2(a_1\cosh M\eta+a_2\sinh M\eta)],\end{aligned} \tag{39}$$

$$\begin{aligned}f_1(\eta)=&\frac{1}{6M\sinh^2M}[7a_1\sinh M+5a_2\cosh M]\sinh M(1-\eta)\\&-\frac{(11a_1+6a_2M)}{6M\sinh^2M}\sinh M\eta+\frac{1}{6M\sinh M}\Big[a_1\cos M(2\eta-1)\\&+a_2\sinh M(2\eta-1)+6a_2M\eta\sinh M(1-\eta)+6Ma_2\eta\cosh M(1-\eta)\\&+3a_2\cosh M(1-\eta)+3a_2\sinh M(1-\eta)+3M^2a_2\eta^2\sinh M(1-\eta)\\&-9(a_2\sinh M-a_1\cosh M)\Big]\end{aligned} \tag{40}$$

where

$$a_1=\frac{1-\cosh M}{2[2(\cosh M-1)-M\sinh M]},\quad a_2=\frac{\sinh M}{2[2(\cosh M-1)-M\sinh M]} \tag{41}$$

$$\left.\begin{aligned} A&=\frac{1}{12M}(98a_1a_2+6M^2a_1a_2+22Ma_1^2+49Ma_2^2) \\ B&=\frac{1}{12M}(24a_1^2-6M^2a_2^2-30Ma_1a_2) \end{aligned}\right\} \tag{42}$$

The expressions for $h_2(\eta)$ and $f_2(\eta)$ and beyond them have not been calculated as they become too cumbersome. .

5. *Discussion*

Let $r=\sqrt{x^2+y^2}$ and the slider be given by $z=d$, $r=a$. Let p_0 be the pressure at the edge of the slider. Then the total lift on the slider is given by

$$L=-\int_0^a\Big[(\tau_{zz})_{z=d}-p_0\Big]2\pi r\ dr \tag{43}$$

and the drag on the slider is given by

$$D=-\int_0^a(\tau_{zx})_{z=d}\ 2\pi r\ dr \tag{44}$$

The expression for τ_{zz} and τ_{zx} for non-Newtonian are given by

$$\tau_{zx}=\rho\frac{UW}{R}f'-\frac{x\rho W^2}{Rd}h''-\rho\alpha[UW(fh''-2hf''+5h'f') \\ +\frac{2xW^2}{d}(3h'h''-hh''')\Big] \tag{45}$$

$$\tau_{zz}=-p+\rho W^2\Big[-\frac{4h'}{R}+8\alpha(2h'^2-hh'')\Big] \tag{46}$$

From the expression for the pressure p, we get

$$-p+p_0=\rho\frac{kW^2}{2d^2}(r^2-a^2) \tag{47}$$

Substituting (45)−(47) and (44) we get the normalized lift L' and drag D' as

$$L'=\frac{L}{(4\nu^2/\pi\rho a^4W^4)}=-\Big[\frac{h'''(0)}{R^3}+\frac{4\alpha\nu^2}{W^2a^2}h''(1)\Big] \tag{48}$$

$$D'=\frac{D}{\rho UW\pi a^2}=-\Big[\frac{f'(1)}{R}+\alpha f''(1)\Big] \tag{49}$$

Taking $\frac{\nu^2}{W^2a^2}=1$ and $\alpha=0{\cdot}1,\ 0{\cdot}15,\ 0{\cdot}2,$ the values of L' and D have been calculated for various values of R by taking $f(\eta)$ and $h(\eta)$ from (21)−(26). These values are given in tables 1 and 2. For hydromagnetic case the expressions for the normalized lift L' and drag D' given by (48) and (49) with $\alpha=0$. Taking $M=1\cdot0$, $1\cdot5$, $2\cdot0$, the values of L' and D' have been calculated by taking $f(\eta)$ and $h(\eta)$ from (37)–(40). These values are given in tables 3 and 4.

Tables 1, 2, 3, and 4 show that the effects of the non-linear terms in the constitutive equation are to increase the lift and to decrease the drag on the slider; thus increase the load bearing capacity of the system. Similar are the effects of a transverse magnetic field when the liquid is Newtonian and electrically conducting. These results are very useful in industry as sliding friction can be greately reduced by using electrically conducting lubricants.

This analogy may be pointed out in the following cases :

1. In couette flow [5] and in a divergent flow between two non-parallel walls [6], the effect of a transverse magnetic field is to flatten the velocity profile in the middle of the channel which is same as due to elastic forces in a similar flow [7].

2. In the flow of Reiner-Rivlin fluid near a stagnation point, Srivastava [8] has shown that the effect of non-Newtonian terms is to reduce the shearing stress on the wall. Kakutani [9] has found that similar is the effect of a transverse magnetic field in the flow of an electrically conducting Newtonian fluid near a stagnation point.

3. Srivastava [10] has solved the flow and the heat transfer of second-order fluid confined between two parallel plates one rotating and other at rest. He has shown that the effects of non-Newtonian terms are to reduce the suction at the stationary disk and to decrease the rate of heat transfer from both the disks. Similar effects have been observed by Srivastava and Sharma by solving similar flow [11] and the heat transfer [12] of a Newtonian fluid under the effect of a transverse magnetic field. Moreover this specific example adds one more geometry to the list in which non-Newtonian and transverse magnetic field effects are similar.

Table I

(Lift for non-Newtonian)

R	$\alpha=0$	$\alpha=0{\cdot}1$	$\alpha=0{\cdot}15$	$\alpha=0{\cdot}2$
0.2	769.446	5634.896	5986.723	6132.124
0.4	98.651	201.464	214.679	222.857
0.6	29.974	55.869	57.864	58.123
0.8	12.964	34.678	35.232	36.123
1.0	6.803	15.436	15.957	16.325

Table II

(Drag for non-Newtonian)

R	$\alpha=0$	$\alpha=0.1$	$\alpha=0.15$	$\alpha=0.2$
0.2	4.718	4 611	4.609	4.593
0.4	2.387	2.070	1.968	1.896
0.6	1.722	1.197	1.175	1.154
0.8	1.474	0.740	0 714	0.695
1.0	1 392	0.453	0.443	0.440

Table III

(Lift for hydro-magnetics)

R	$M=0$	$M=1{\cdot}0$	$M=1{\cdot}5$	$M=2{\cdot}0$
0.2	769.446	3998.689	4320.586	4732.681
0.4	98.651	111.682	176.855	190.454
0.6	29.974	45.390	49.562	52 832
0.8	12.964	25.678	27.554	28.032
1.0	6 803	8.123	8.153	8.183

Table IV

(Drag for hydro-magnetics)

R	$M=0$	$M=1{\cdot}0$	$M=1{\cdot}5$	$M=2{\cdot}0$
0.2	4.718	4.697	4.603	4.570
0.4	2.387	2.235	2.199	2.163
0.6	1.722	1.579	1.123	0.984
0.8	1.474	1.223	0.904	0.814
1.0	1.392	1.119	0 747	0.673

References

1. B. D. Coleman and W. Noll, An Approximation Theorem for functionals with applications in continuous mechanics, Arch. Mech. Anal. 6,355 (1960).

2. C. Y. Wang, Fluid Dynamics of the circular Porous slider, Jour, Appl. Mech. Trans. ASME. 41,2,343 (1974).

3. H. Markovitz and B. D. Coleman Adv. Appl. Mech. 8, 86 (1964).

4. C. Truesdell, the Natural time of a visco-elastic fluid. Its significance and measurement. Phy. Fluids, 7,1134 (1964).

5. T. G. Cowling, Magneto-hydronamics, Interesci, Publ. Inc. New York (1957).

6. A. C. Srivastava, the effect of magnetic field on the Flow between two non-parallel Planes. Proc. 5th Cong. Theo. Appl. Mech. Roorkee, India, Dec. 1959.

7. S. K. Sharma, Visco-elastic steady flow, Zeit. Angew. Math. U.Mach. 3,39,313 (1959).

8. A. C. Srivastava, The flow of a non-Newtonian liquid near a stagnation point. Zeit. Angew. Math. U.Phys. 9, Fasc. 1.80 (1958).

9. T. Kakutani, Axially Symmetri stagnation-point Flow of an electrically conducting Fluid under Transverse Magnetic Field. J.Phys. Soc. Japan, 4,688 (1960).

10. A. C. Srivastava, Flow of Second-order Fluids with Heat Transfer, Amer. Inst. Aero. Astro. Journal, 2. No. 4.750 (1964).

11. A. C. Srivastava and S. K. Sharma, the effect of a transverse magnetic field on the Flow between two infinite disks—one rotating and other at rest. Bull. Acad. Pol. Sci. Ser. Sci. Tech. 9,11,639 (1961).

12. A. C. Srivastava and S. K. Sharma, Heat Transfer due to the Flow between two infinite plates (one rotating and the other at rest) under a Transverse magnetic field. Jour. Phy. Soc. Japan, 19,8,1390 (1964).