**Mathematical Forum, Vol II, No. 1, March 1979, pp. 32-38**

# ON THE STABILITY OF TWO SUPERPOSED FLUIDS IN THE PRESENCE OF SUSPENDED PARTICLES

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( Received on 13.6.78 )

Revised on 10.12.78

ABSTRACT

The instability of the plane interface between two superposed viscous fluids in the presence of suspended particles is studied. The dispersion relation is obtained. The stability analysis has been carried out for two highly viscous fluids of equal kinematic viscosities, for mathematical simplicity. The system is found to be stable for stable configuration and unstable for unstable configuration in the absence or presence of suspended particles. This is in contrast to the thermal instability (Bénard convection) problem where the suspended particles have a destabilizing effect.

## 1. *Introduction*

The stability of superposed fluids under varying assumptions of hydrodynamics has been discussed in a treatise by Chandrasekhar (1961). Chandra (1938) observed a contradiction between the theory for onset of convection in fluids heated from below in his experiment. He performed the experiment in an air layer and found that the instability depended on the depth of the layer. A Bénard-type cellular convection with fluid descending at the cell centre was observed when the predicted gradients were imposed, for layers deeper than 10 mm. A convection which was different in character from that in deeper layers occurred at much lower gradients than predicted, if the layer depth was less than 7 mm and called this motion columnar instability. He added an aerosol to mark the flow pattern. Motivated by interest in fluid-particle mixtures and columnar instability, Scanlon and Segel (1973) considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. Sharma et. al. (1976) studied the effect of suspended particles on the onset of Bénard convection in hydromagnetics. The effect of suspended particles was found to destabilize the layer whereas the effect of magnetic field was stabilizing. The effect of a uniform rotation was also studied and was found to have a stabilizing effect in the presence of suspended particles on the Bénard convection. Sharma (1975) studied the gravitational instability problem in the presence of suspended particles. Sharma (1977) also studied the magneto-gravitational instability in the presence of suspended particles. It is found that Jeans' criterion determines the gravitational instability.

The effect of suspended particles on the stability of superposed fluids might be of industrial and chemical engineering importance. Further motivation for this study is the fact that knowledge concerning fluid-particle mixture is not commensurate with their industrial and scientific importance. In the present paper, a study has been made of the stability of two superposed fluids in the presence of suspended particles.

## 2. *Perturbation Equations*

Consider a static state in which an incompressible fluid-particle layer of variable density is arranged in horizontal strata and the pressure $p$ and the density $\rho$ are functions of the vertical coordinate $z$ only. The character of the equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then following its further evolution.

Let $\rho$, $\mu$, $p$ and $\mathbf{u}(u, v, w)$ denote respectively the density, the viscosity, the pressure and the velocity of the pure gas; $\mathbf{V}(\mathbf{x}, t)$ and $N(\mathbf{x}, t)$ denote the velocity and number density of the particles respectively. $K=6\pi\mu\eta$ where $\eta$ is the particle radius, is a constan $\mathbf{x}=(x, y, z)$ and $\underline{\lambda}=(0, 0, 1)$. Then the equations of motion and continuity for the gas are

$$\rho\left[\frac{\partial u}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p-\rho g\lambda+\mu\nabla^2\mathbf{u}+\left(\frac{\partial w}{\partial \mathbf{x}}+\frac{\partial \mathbf{u}}{\partial z}\right)\frac{d\mu}{dz}+KN(\mathbf{V}-\mathbf{u}), \tag{1}$$

$$\nabla.\,\mathbf{u}=0, \tag{2}$$

In the equations of motion (1) the presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid. Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid; there must be an extra force term, equal in magnitude but opposite in sign, in the equation of motion for the particles. The buoyancy force on the particles is neglected. Interparticle reactions are also not considered for we assume that the distances between particles are quite large compared with their diameter.

The equations of motion and continuity, under the above assumptions, are

$$mN\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\nabla)\mathbf{V}\right]=-mNg\lambda+KN(\mathbf{u}-\mathbf{V}), \tag{3}$$

$$\frac{\partial N}{\partial t}+\nabla.(N\mathbf{V})=0, \tag{4}$$

where $mN$ is the mass of particles per unit volume and $g$ is the acceleration of gravity.

Let $\delta\rho$ and $\delta p$ denote respectively the perturbations in density $\rho$ and pressure $p$. Then the linearized perturbation equations of the fluid-particle layer are :

$$\rho\frac{\partial \mathbf{u}}{\partial t}+-\nabla\delta p-g\delta\rho\lambda+\mu\nabla^2\mathbf{u}+KN(\mathbf{V}-\mathbf{u})+\left(\frac{\partial w}{\partial \mathbf{x}}+\frac{\partial u}{\partial \tau}\right)\frac{d\mu}{d\tau}, \tag{5}$$

$$\triangle.\mathbf{u}=0, \tag{6}$$

$$\left(\frac{m}{K}\frac{\partial}{\partial t}+1\right)\mathbf{V}=\mathbf{u} \tag{7}$$

In addition to equations (5)-(7), we have the equation

$$\frac{\partial}{\partial t}\delta\rho=-w\left(\frac{d\rho}{d\tau}\right), \tag{8}$$

which ensures that the density of every particle remains unchanged as we follow it with its motion.

We analyse the disturbance into normal modes and seek solutions whose dependence on $x$, $y$ and $t$ is of the form

$$f(z)\exp\left(ik_x x+ik_y y+nt\right), \tag{9}$$

where $k=(k^2{}_x+k^2{}_y)^{\frac{1}{2}}$ is the wave number of the disturbance and $n$ is the growth rate of perturbation. Using expression (9), equations (5) and (6) with the help of (7) and (8) become

$$[\rho(\tau n+1)+mN]nu=-(1+\tau n)ik_x\delta p+\mu(D^2-k^2)(1+\tau n)u$$
$$+(ik_x w+Du)(1+\tau n)D\mu, \tag{10}$$

$$[\rho(\tau n+1)+mN]nv=-(1+\tau n)ik_y\delta p+\mu(1+\tau n)(D^2-k^2)v$$
$$+(ik_y w+Dv)(1+\tau n)D\mu, \tag{11}$$

$$[\rho(\tau n+1)+mN]nw=-(1+\tau n)D\delta p+\mu(1+\tau n)(D^2-k^2)w$$
$$+2Dw(1+\tau n)D\mu+\frac{g}{n}(D\rho)(1+\tau n), \tag{12}$$

$$ik_x u+ik_y v+Dw=0, \tag{13}$$

where $D=d/d\tau$.

Eliminating $\delta p$ between equations (10)-(12) and using equation (13), we obtain

$$n(\tau n+1)[D(\rho Dw)-k^2\rho w]+[D(mN\,Dw)-k^2(mN)w]$$
$$-\mu(1+\tau n)(D^2-k^2)^2w+\frac{gk^2}{n}(D\rho)(1+\tau n)w \tag{14}$$
$$-(1+\tau n)[D\{(D\mu)(D^2+k^2)w\}-2k^2(D\mu)(Dw)]=0.$$

3. *Two Uniform Fluids Separated by a Horizontal Boundary*

Let two uniform fluids of densities $\rho_1$, $\rho_2$ and viscosities $\mu_1$, $\mu_2$ be separated by a horizontal boundary at $\tau=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids, respectively. In each of the two regions of constant $\rho$ and $\mu$, equation (14) reduces to

$$(D^2-k^2)(D^2-K'^2)w=0, \tag{15}$$

where $K'^2=k^2+\frac{n}{\nu}+\frac{mN}{\mu(1+\tau n)}$ and $\nu=\mu/\rho$ is the kinematic viscosity.

Since $w$ must vanish both when $\tau\to-\infty$ ( in the lower fluid ) and $\tau\to+\infty$ ( in the upper fluid ), the general solution of equation (15) can be written as

$$w_1=A_1e^{+k\tau}+B_1e^{+K_1\tau}, \; (\tau<0) \tag{16}$$

$$w_2=A_2e^{-k\tau}+B_2e^{-K_2\tau}, \; (\tau>0) \tag{17}$$

where $A_1$, $B_1$, $A_2$, $B_2$ are constants of integration,

$$K_1=\sqrt{k^2+\frac{n}{\nu_1}+\frac{mN}{\rho_1\nu_1(1+\tau n)}}, \tag{18}$$

and
$$K_2=\sqrt{k^2+\frac{n}{\nu_2}+\frac{mN}{\rho_2\nu_2(1+\tau n)}}. \tag{19}$$

Integrating equation (14) across the interface at $\tau=0$, we obtain

$$\left\{\left[\rho_2-\frac{\mu_2}{n}(D^2-k^2)\right]Dw_2\right\}_{z=0}-\left\{\left[\rho_1-\frac{\mu_1}{n}(D^2-k^2)\right]Dw_1\right\}_{\tau=0}$$

$$+\frac{mN}{n(\tau n+1)}(Dw_2-Dw_1)_{\tau=0}=-\frac{gk^2}{n^2}(\rho_2-\rho_1)w_0$$

$$-\frac{2k^2}{n}(\mu_2-\mu_1)(Dw)_0 \tag{20}$$

In addition to the condition (20), the boundary conditions to be satisfied at the interface $\tau=0$ are (Chandrasekhar, 1961, p. 432) : $W$, $Dw$ and $\mu(D^2+K^2)w$ must be continuous across an interface between two fluids. (21)

Applying the boundary conditions (20) and (21) to the solutions given in (16) and (17), we obtain

$$A_1+B_1=A_2+B_2, \; (=w_0), \tag{22}$$

$$kA_1+K_1B_1=-kA_2-K_2B_2, \; (=Dw_0), \tag{23}$$

$$\mu_1[2k^2A_1+(K_1{}^2+k^2)B_1]=\mu_2[2k^2A_2+(K_2{}^2+k^2)B_2],$$
$$(=\mu(D^2+k^2)w_0), \tag{24}$$

and $$\left[\frac{R}{2}+C-\rho_1-\frac{mN}{n(\tau n+1)}\right]A_1+\left[\frac{R}{2}+C\frac{K_1}{k}\right]B_1 \tag{25}$$
$$+\left[\frac{R}{2}-C-\rho_2-\frac{mN}{n(\tau n+1)}\right]A_2+\left[\frac{R}{2}-C\frac{K_2}{k}\right]B_2=0,$$
where $R=\frac{gk}{n^2}(\rho_2-\rho_1)$ and $C=\frac{k^2}{n}(\mu_2-\mu_1)$.

Equations (22)-(25) can be written, in matrix notation, in the form of the single matrix equation

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ k & K_1 & 2k & K_2 \\ 2k^2\mu_1 & \mu_1(k_1{}^2+k^2 & -2k^2\mu_2 & -\mu_2(k_2{}^2+k^2) \\ \left\{\frac{R}{2}+C-\rho_1-\frac{mN}{n(\tau n+1)}\right\} & \left\{\frac{R}{2}+C\frac{K_1}{k}\right\} & \left\{\frac{R}{2}-C-\rho_2-\frac{mN}{n(\tau n+1)}\right\} & \left(\frac{R}{2}-C\frac{K_2}{k}\right) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix}=0. \tag{26}$$

The determinant of the linear system of equations which (26) represents must clearly vanish. The determinant can be reduced by subtracting the first column from the second, the third column from the fourth and adding the first column to the third. By this procedure, we obtain

$$\begin{vmatrix} K_1-k & 2k & K_2-k \\ \rho_1 n+\frac{mN}{\tau n+1} & 2k^2(\rho_1\nu_1-\rho_2\nu_2) & -\left(\rho_2 n+\frac{mN}{\tau n+1}\right) \\ \rho_1+\frac{mN}{n(\tau n+1)}+\frac{C}{k}(K_1-k) & \left[\frac{gk}{n^2}(\rho_2-\rho_1)-(\rho_1+\rho_2)-2\frac{mN}{n(\tau n+1)}\right] & \left[\rho_2+\frac{mN}{n(\tau n+1)}-\frac{C(K_2-k)}{k}\right] \end{vmatrix}=0. \tag{27}$$

## 4. *Dispersion Relation and Discussion*

The dispersion relation (27) is quite complicated as the values of $K_1$ and $K_2$ involve square roots. To simplify this, we make the assumptions that the kinematic viscosities of the two fluids are the same i.e., $\nu_1=\nu_2=\nu$ and that the fluids are highly viscous. Under the above approximations, we have

$$K'=k\left[1+\frac{n}{\nu k^2}+\frac{mN}{\rho\nu k^2(\tau n+1)}\right]^{\frac{1}{2}}=k\left[1+\frac{n}{2\nu k^2}+\frac{mN}{2\rho\nu k^2(\tau n+1)}\right], \tag{28}$$

So that

$$K_1-k=\frac{n}{2\nu k}+\frac{mN}{2\rho_1\nu k(\tau n+1)}. \tag{29}$$

and
$$K_2-k=\frac{n}{2\nu k}+\frac{m}{2\rho_2\nu k(\tau n+1)} \tag{30}$$

Substituting the values of $K_1—k$ and $K_2—k$ from equations (29) and (30) in the determinant (27) and simplifying it, after a little algebra, we obtain

$$A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0=0, \tag{31}$$

where

$$\begin{aligned}
A_7&=\rho_1\rho_2(\rho_1+\rho_2)^2\tau^3,\\
A_6&=2\nu k^2\rho_1\rho_2(\rho_1+\rho_2)^2z^3+3\rho_1\rho_2(\rho_1+\rho\ )^2\tau^2,\\
A_5&=6\nu k^2\rho_1\rho_2\tau^2(\rho_1+\rho_2)^2+gk\rho_1\rho_2(\rho_1+\rho_2)(\rho_1-\rho_2)\tau^3\\
&\quad+3\rho_1\rho_2\tau(\rho_1+\rho_2)^2+mN\tau^2(\rho_1+\rho_2)(\rho_1{}^2+4\rho_1\rho_2+\rho_2{}^2),\\
A_4&=6\nu k^2\tau\rho_1\rho_2(\rho_1+\rho_2)^2+2\nu k^2mN\tau^2(\rho_1+\rho_2)^3+3gk\tau^2\rho_1\rho_2\\
&\quad(\rho_1-\rho_2)(\rho_1+\rho_2)+2mN\tau(\rho_1+\rho_2)(\rho_1{}^2+4\rho_1\rho_2{}^2)+2\rho_1\rho_2(\rho_1+\rho_2),\\
A_3&=4\nu k^2mN\tau(\rho_1+\rho_2)^3+2\nu k^2\rho_1\rho_2(\rho_1+\rho_2)^2+3gk\rho_1\rho_2\tau\\
&\quad(\rho_1+\rho_2)(\rho_1-\rho_2)+mN(\rho_1+\rho_2)(\rho_1{}^2+4\rho_1\rho_2+\rho_2{}^2)+gk(\rho_1-\rho_2)\\
&\quad(\rho_1+\rho_2)^2mN\tau^2+3m^2N^2\tau(\rho_1+\rho_2)^2, \qquad (32)\\
A_2&=2\nu k^2m^2N^2\tau(\rho_1+\rho_2)^2+2gkmN\tau(\rho_1-\rho_2)(\rho_1+\rho_2)^2+gk\rho_1\rho_2\\
&\quad(\rho_1-\rho_2)(\rho_1+\rho_2)+3m^2N^2(\rho_1+\rho_2)^2+2\nu k^2mN(\rho_1+\rho_2)^3,\\
A_1&=gkmN(\rho_1-\rho_2)(\rho_1+\rho_2)^2+gk(\rho_1-\rho_2)(\rho_1+\rho_2)m^2N^2\tau\\
&\quad+2m^3N^3(\rho_1+\rho_2)+2\nu k^2m^2N^2(\rho_1+\rho_2)^2,\\
A_0&=gkm^2N^2(\rho_1-\rho_2)(\rho_1+\rho_2).
\end{aligned}$$

For the potentially stable arrangement $\rho_1>\rho_2$, we find that all the coefficients in (31) are positive and so all the roots of $n$ are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest, negative real roots. The system is therefore stable in each case. Thus the potentially stable configuration remains stable even in the presence of suspended particles on the stability of superposed fluids.

For the unstable configuration $\rho_2>\rho_1$, there is at least one change of sign in equation (31) and so equation (31) has one positive root. The occurrence of positive root implies that the system is unstable. The unstable configuration, therefore, remains unstable for the case of two superposed fluids in the presence of suspended particles.

We conclude therefore that the system is stable for stable configuration and unstable for unstable configuration in the absence [Chandrasekhar (1961), Chapter X] or presence of suspended particles. This is in contrast to the thermal instability (Bénard convention) where the suspended particles have a destabilizing effect.

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