VISCOUS CHANNEL FLOW INDUCED BY PRESCRIBED INJECTION

S. DATTA

Department of Mathematics and Astronomy, Lucknow University
Lucknow (INDIA)

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ABSTRACT

In the present paper we have studied the flow of an incompressible viscous fluid induced by prescribed injection in a straight two dimensional channel just before it gets fully developed. Two cases have been discussed : (i) when the injection velocity varies as the inverse square of a linear function of the distance from the inlet and (ii) when it decreases exponentially with this distance. In the first case analysis is applicable for all values of injection Reynolds number but in the second case we have to distinguish between the low and high value situations. For investigating, the latter matching technique has been employed. Expressions for the centre section velocity have been obtained in different situations.

1. *Introduction*

Study of fluid flow in pipes and channels is a classical problem. Extensive literature on the fully developed flow as well as on the flow in inlet region exists. In the present paper we have followed a different approach to investigate the flow in a channel ; instead of developing the flow from the prescribed entrance velocity we proceed to determine the flow picture just before it gets fully developed.

Let us consider an infinite straight channel filled with an incompressible viscous fluid initially at rest. The flow is generated by fluids injection normal to the channel's boundary in a prescribed manner. In order that the fluid does not get accumulated it will be carried away in the axial direction. After certain time a steady state is reached which will be discussed here. The flow picture is complicated near the entrance but gets simplified as we move away from the entrance, ultimately emerging as the well known parabolic profile. Therefore, we shall endeavour to construct the pattern backward starting from the parabolic form. Two cases will be considered : (i) the injection velocity varies as the inverse square of a linear function of the distance from the entrance and (ii) it decreases exponentially with this distance.

2. *Basic Equations :*

The flow is governed by the Navier Stokes equations together with the continuity equation which in the present two-dimensional situation in the usual notation can be expressed as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad \dots (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad \dots (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad \dots (3)$$

We shall take the x-axis along the midsection of the channel which is of width $2h$ and y-axis perpendicular to it at the inlet, alternately, we can consider the right half of a doubly infinite channel with the injection and subsequent flow symmetric about the y-axis. Of course the flow is symmetric about the x-axis too.

Case (i). The injection velocity $v_i=\mp q_o\frac{h}{(h+\alpha x)^2}$. Where the $\mp$ signs correspond to $y=\pm h$. Here q_o is the velocity at $x=0$ and α a dimensionless parameter.

Let us introduce the non-dimensional variables

$$\xi=\frac{h}{h+\alpha x} \quad \text{and} \quad \eta=\frac{y}{h},$$

and use the expansions

$$\frac{u}{q_o}=u_o(\eta)+\xi u_1(\eta)+\xi^2 u_2(\eta)+\xi^3 u_3(\eta)+\dots,$$

$$\frac{v}{q_o}=\xi^2 v_2(\eta)+\xi^3 v_3(\eta)+\dots,$$

$$\frac{p}{\frac{\rho\nu q_o}{h}}=\frac{p_{-1}(\eta)}{\xi}+p_l(\eta)\log\xi+p_o(\eta)+p_1(\eta)\xi+p_2(\eta)\xi^2+\dots,$$

to get from equations (1), (2) and (3) on equating powers of the following sets of equations

$$\frac{d^2 u_0}{dy^2}=\alpha p_{-1}, \qquad \dots (1a)$$

$$\frac{d^2 u_1}{dy^2}=-\alpha p_l, \qquad \dots (1b)$$

$$\frac{d^2 u_2}{dy^2}=-\alpha p_1+\mathrm{Re}\left[-\alpha u_0 u_1+v_2\frac{du_0}{dy}\right], \qquad \dots (1c)$$

$$\frac{d^2u_3}{dy^2}=-2\alpha p_2-2\alpha^2u_1+\mathrm{Re}\left[-\alpha u_1{}^2-2\alpha u_0u_2+v_2\frac{du_1}{d\eta}\right.$$

$$\left.+v_3\frac{du_0}{d\eta}\right]; \qquad \text{... (1d)}$$

$$\frac{dp_{-1}}{d\eta}=\frac{dp_l}{d\eta}=\frac{dp_0}{d\eta}=\frac{dp_1}{d\eta}=0, \qquad \text{... (2a)}$$

$$\frac{d^2v_2}{d\eta^2}=\frac{dp_2}{d\eta} \qquad \text{... (2b)}$$

$$\frac{d^2v_3}{d\eta^2}=\frac{dp_3}{d\eta}-\mathrm{Re}\ 2\alpha u_0v_2; \qquad \text{... (2c)}$$

and

$$\frac{du_2}{d\eta}=\alpha u_1, \qquad \text{... (3a)}$$

$$\frac{dv_3}{d\eta}=2\alpha u_2; \qquad \text{... (3b)}$$

where $Re=\frac{q_0h}{\nu}$ is the injection Reynolds number.

Equation (2a) show that p_{-1}, p_l, p_0 and p_1 are constants and so the solutions of (1a) and (1b) satisfying the no slip conditions at $\eta=\pm1$ are

$$u_0=-\frac{\alpha}{2}p_{-1}(1-\eta^2), \qquad \text{... (4a)}$$

$$u_1=\frac{\alpha}{2}p_l(1-\eta^2). \qquad \text{... (4b)}$$

Also, since v_2 vanishes at $\eta=0$, we have from (3a)

$$v_2=\alpha\int_0^1 u_1\ dy=\frac{\alpha^2}{2}p_l\left(\eta-\frac{\eta^3}{3}\right) \qquad \text{... (4c)}$$

Now p_{-1} is obtained by applying the condition that the flow at infinity ($\xi=0$) equals total fluid injected. Thus, we have

$$c_0h\int_{-1}^{1}u_0d\eta=2q_0\int_0^{\infty}\frac{h^2}{(h+\alpha\eta)^2}dx$$

giving

$$p_{-1}=-\frac{3}{\alpha} \qquad \text{... (5a)}$$

Again, using the condition $v_2=-1$ at $\eta=1$, we have

$$p_1=-\frac{3}{\alpha}. \qquad \text{... (5b)}$$

Substituting the values of u_0, u_1 and v_2 in equation (1c), we get

$$\frac{d^2 u_2}{d\eta^2} = -\alpha p_1 + \frac{9}{4}\frac{Re}{\alpha}\left(1+\frac{\eta^4}{3}\right).$$

whose solution satisfying $u_2=0$ at $\eta=\pm 1$ is

$$u_2 = -\frac{9}{4}\frac{Re}{\alpha}\left(\frac{1-\eta^2}{2}+\frac{1-\eta^6}{90}\right)+\frac{\alpha}{2}p_1(1-\eta^2).$$

Further the condition $\int_{-1}^{1} u_2 d\eta = 0$ gives

$$p_1 = \frac{81}{35}\frac{Re}{\alpha}$$

Hence

$$u_2(\eta) = \frac{9}{40}\frac{Re}{\alpha}\left[\frac{1-\eta^2}{7}-\frac{1-\eta^6}{9}\right], \tag{6a}$$

and so from (3b)

$$\begin{aligned} v_3 &= 2\alpha \int_0^{\eta} u_2(\eta) d\eta \\ &= \frac{3}{20} Re \left[\frac{2}{21}\eta - \frac{\eta^3}{7} + \frac{\eta^7}{21}\right]. \end{aligned} \tag{6b}$$

Integrating equation (2b), we have

$$p_2 = \frac{dv_2}{d\eta} + \pi_2 = \alpha u_1 + \pi_2$$

Substituting the values of u_0, u_1, u_2, v_2, v_3 and p_2 as obtained above in equation (1d), we get

$$\begin{aligned} \frac{d^2 u_3}{d\eta^2} &= -2\pi_2\alpha + 6\alpha(1-\eta^2) \\ &\quad -\frac{9}{4}\frac{Re}{\alpha}\left[1+\frac{\eta^4}{3}+\frac{1}{10}Re\left\{\frac{2}{21}-\frac{\eta^2}{3}+\frac{\eta^4}{7}+\frac{\eta^6}{3}-\frac{5}{21}\eta^8\right\}\right]. \end{aligned}$$

The solution of the above equation satisfying the conditions

$$u_3 = 0 \text{ at } = 1 \text{ and } \int_{-1}^{1} u_3 d\eta = 0 \text{ is}$$

$$\begin{aligned} u_3 &= \alpha\left[-\frac{1}{10}+\frac{3\eta^2}{5}-\frac{\eta^4}{2}\right]+\frac{9\,Re}{4\alpha}\left[-\frac{1}{315}+\frac{\eta^2}{70}-\frac{\eta^6}{90}\right. \\ &\quad \left.+\frac{Re}{10}\left\{\frac{1847}{582120}-\frac{37}{1617}\eta^2+\frac{\eta^4}{36}-\frac{\eta^6}{210}-\frac{\eta^8}{168}+\frac{\eta^{10}}{378}\right\}\right]. \end{aligned} \tag{7}$$

The centre section velocity when $\alpha = Re = I$ is now given by

$$\frac{u(0)}{q_0} = \frac{3}{2}(1-\xi)+\frac{1}{140}\xi^2-\frac{275353}{2587200}\xi^3+\dots;$$

the above expression shows how for small values of ξ the value decreases from the fully developed value $\frac{3}{5}$. The analysis can be extended in a similar way to obtain further terms to increase the range of applicable values of ξ.

Case (ii). In this case injection velocity is expressed by

$$v_i = \mp q_0 e^{\frac{-\alpha a}{h}} \text{ at } y = \pm h.$$

let us put

$$\xi = e^{\frac{-\alpha a}{h}}, \quad \eta = \frac{y}{h},$$

$$\frac{u}{q_0} = u_0(\eta) + \xi u_1(\eta) + \xi^2 u_2(\eta) + \ldots,$$

$$\frac{v}{q_0} = \xi v_1(\eta) + \xi^2 v_2(\eta) + \ldots,$$

$$\frac{p}{\frac{\rho\nu q_0}{h}} = p_l(\eta) \log \xi + p_0(\eta) + p_1(\eta)\xi + p_2(\eta)\xi^2 + \ldots,$$

Substituting these in the flow equations (1), (2) and (3), we get on equating powers of ξ the following sets of equations

$$\frac{d^2 u_0}{d\eta^2} = -\alpha p_l, \qquad \ldots \text{(9a)}$$

$$\frac{d^2 u_1}{d\eta^3} + \alpha^2 u_1 = -\alpha p_1 - \text{Re}\left[\alpha u_0 u_1 - v_1 \frac{du_0}{d\eta}\right]; \qquad \ldots \text{(9b)}$$

$$\frac{dp_1}{d\eta} = \frac{dp_0}{d\eta} = 0, \qquad \ldots \text{(10a)}$$

$$\frac{d^2 v_1}{d\eta^2} + \alpha^2 v_1 = \frac{dp_1}{d\eta} - \text{Re}\, \alpha u_0 v_1, \qquad \ldots \text{(10b)}$$

and

$$\frac{dv_1}{d\eta} = \alpha u_1, \quad \frac{dv_2}{d\eta} = 2\alpha u_2, \quad \frac{dv_3}{d\eta} = 3\alpha u_3. \qquad \ldots \text{(11)}$$

Equation (10a) show that p and p_0 are constants. Now as in case (i) the solution of equation (9a), satisfying the no slip condition and the condition that the flow at infinity equals total amount of fluid injected, is

$$u_0 = \frac{3}{2\alpha}(1-\eta^2). \qquad \ldots \text{(I2)}$$

Further eliminating p_1 and u_1 in between (9b), (10b) and (11), we get the following equation for v_1

$$\frac{d^4 v_1}{d\eta^4} + 2\alpha^2 \frac{d^2 v_1}{d\eta^2} + \alpha^4 v_1 + \frac{3}{2}\text{Re}\,(1-\eta^2)\frac{d^2 v_1}{d\eta^2}$$
$$+ (2 + \alpha^2 - \alpha^2\eta^2) v_1 = 0 \qquad \ldots \text{(13)}$$

As a solution of this fourth order equation is not easy to obtain we shall determine approximations for small and high Renolds number.

Approximation when Re is small.

A regular perturbation expansion

$$v_1 = v_{10} + \text{Re}\, v_{11} + \text{Re}^2 v_1 + \ldots$$

provides

$$\frac{d^4 v_{10}}{d\eta^4} + 2\alpha^2 \frac{d^2 v_{10}}{d\eta^2} + \alpha^4 v_{10} = 0\,, \qquad \ldots \text{(14a)}$$

$$\frac{d^4 v_{11}}{d\eta^4} + 2\frac{2d^2 v_{11}}{d\eta^2} + \alpha^4 v_{11} = -\tfrac{3}{2}(1-\eta^2)\frac{d^2 v_{10}}{d\eta^2}$$

$$+ (2 + \alpha^2 - \alpha^2\eta^2) v_{10}\,, \qquad \ldots \text{(14b)}$$

$$\frac{d^4 v_{12}}{d\eta^4} + 2\alpha^2 \frac{d^2 v_2}{d\eta^2} + \alpha^4 v_{12} = -\tfrac{3}{2}[(1+\eta^2)\frac{d^2 v_{11}}{d\eta^2}$$

$$+ (2 + \alpha^2 - \alpha^2\eta^2) v_{11}] \qquad \ldots \text{(14c)}$$

These have to be solved under the boundary conditions

$$v_{10}(1) = -1,\ v_{10}(-1) = 1,\ v'_{10}(\pm 1) = 0,$$

$$v_{11}(\pm 1) = v'_{11}(\pm 1) = v_{12}(\pm 1) = v'_{12}(\pm 1) = 0 \qquad \ldots \text{(15)}$$

Solution of (14a) satisfying the relevant conditions in (15) is

$$v_{10} = \frac{1}{\sin\alpha \cos\alpha - \alpha}[(\alpha \sin - \cos\alpha) \sin \alpha\eta$$

$$+ (\alpha \cos\alpha)\eta \cos \alpha\eta], \qquad \ldots \text{(16a)}$$

and so

$$v_{10} = \frac{1 d v_{10}}{d\eta}$$

$$= \frac{1}{\alpha(\sin\alpha \cos\alpha - \alpha)}[(\alpha \sin\alpha - \cos\alpha)\, \alpha \cos \alpha\eta$$

$$+ \alpha \cos \alpha\eta - \sin \alpha\eta)]. \qquad \ldots \text{(16b)}$$

Substituting the value of v_{10} in equation (14b) and then solving it we get

$$v_{11} = C \sin \alpha\eta + D\eta \cos \alpha\eta$$

$$+ \frac{3}{u\alpha^2(\sin\alpha \cos\alpha - \alpha)}[(\alpha \sin\alpha - \cos\alpha - 7\alpha^2 \cos\alpha)$$

$$+ \frac{\eta^2}{2\alpha^2} \sin \alpha\eta + \tfrac{1}{2}(\alpha \cos\alpha)\eta^3 \cos \alpha\eta + \frac{\alpha^2 \cos\alpha}{12}\eta^4 \sin \alpha\eta],$$

$$\ldots \text{(17a)}$$

and so

$$u_{11}=\frac{1}{\alpha}\frac{dv_{11}}{\alpha d\eta}=C\cos\alpha\eta+\frac{D}{\alpha}(\cos\alpha\eta-\alpha\eta\sin\alpha\eta)$$

$$+\frac{3}{4\alpha^3(\sin\alpha\cos\alpha-\alpha)}[(\alpha\sin\alpha-\cos\alpha-7\alpha^2\cos\alpha)\frac{\eta}{\alpha^2}\sin\eta$$

$$+(\alpha\sin\alpha-\cos\alpha-4\alpha^2\cos\alpha)\frac{\eta^2}{2\alpha}\cos\alpha\eta-\frac{\alpha^2}{6}\cos\alpha\eta^3$$

$$\sin\alpha\eta+\frac{\alpha^3}{12}\cos\alpha\eta^4\cos\alpha\eta], \quad \ldots \text{(17b)}$$

where

$$C=\frac{-3}{4\alpha\sin\alpha(\sin\alpha\cos\alpha-\alpha)^2}\left[\frac{\alpha}{2}\cos^4\alpha+\sin\alpha\cos^3\alpha\right.$$

$$\left(\frac{\alpha^2}{12}-\frac{3}{2}-\frac{1}{12\alpha}+\frac{1}{2\alpha^2}\right)+\sin^2\alpha\cos^2\alpha\left(\frac{\alpha}{4}+\frac{4}{\alpha}-\frac{1}{\alpha^3}\right)-\frac{1}{2\alpha^2}$$

$$\left.\cos\alpha\sin^2\alpha-\left\{\frac{\sin^2\alpha}{2\alpha}+\frac{\alpha}{9}\cos^2\alpha+\sin\alpha\cos\alpha\left(\frac{\alpha^2}{12}-4\right)\right\}\right]$$

and

$$D=\frac{-3}{4\alpha(\sin\alpha\cos\alpha-\alpha)^2}\left[\frac{\sin^3\alpha}{\alpha^2}-\sin^2\alpha\cos\alpha\left(\frac{\alpha}{6}+\frac{8}{\alpha}+\frac{1}{\alpha^3}\right)\right.$$

$$\left.=\sin\alpha\cos^2\alpha\left(\frac{1}{12\alpha}+2-\frac{1}{2\alpha^2}-\frac{\alpha^2}{12}\right)-\frac{\alpha}{2}\cos^3\alpha\right].$$

For $\alpha=1$ the centre line velocity is given by

$$\frac{u(0)}{q_0}=\frac{3}{2}+\xi\, Re\, u_{11}(0)$$

$$=\frac{3}{2}-0.057\,\xi\, Re.$$

This shows how $u(0)$ decreases as we move backward from the fully deveioped region. The accuracy can be again increased by calculating further terms in a similar manner.

Approximation for large Re.

Since the small parameter 1/Re multiplies the highest derivative a regular perturbation in powers of 1/Re is not tenable. Instead we determine two approximations one away from the boundary and the other near the boundary and then match them to determine the undetermined constants. As the flow is symetrical about the centre section with v_1 an odd function we can consider only the region $0\leqslant\eta\leqslant 1$.

Away from the boundary and near $\eta=0$ the following approximation of equation (13), obtained by retaining the dominent terms, is tenable for high Reynaids number

$$\frac{d^4 v_1}{d\eta^4}+\frac{3}{2}\ \text{Re}\,(1-\eta^2)\frac{d^2 v_1}{d\eta^2}=0 \qquad \cdots(19)$$

with $\frac{d^2 v_1}{d\eta^2}=\forall$ this becomes

$$\frac{d^2\forall}{d\eta^2}+\frac{3}{2}\ \text{Re}\,(1-\eta^2)\forall=0$$

Now *WKB* approximation [2] of the above equation provides

$$\frac{d^2 v_1}{d\eta^2}=v=k_1\ \frac{\sin\{\sqrt{\tfrac{3}{2}}\ \text{Re}\,[\tfrac{1}{2}\eta\,(1-\eta^2)^{\frac{1}{2}}+\tfrac{1}{2}\sin^{-1}\eta\}]}{(1-\eta^2)^{\frac{1}{4}}} \qquad \ldots\ (20a)$$

where the odd character of $\frac{d^2 v_1}{d\eta^2}$ has been taken into consideration and k_1, is a constant still to be determined.

Intergrating the above equation, we get

$$\frac{dv_1}{d\eta}=k\int_0^1\frac{\sin\sqrt{\tfrac{3}{2}}\ \text{Re}\,[\tfrac{1}{2}\eta(1-\eta^2)^{\frac{1}{2}}+\tfrac{1}{2}\sin^{-1}\eta]\}}{(1-\eta^2)^{\frac{1}{4}}}d\eta+k_8 \qquad \ldots\ (20\ b)$$

and

$$v_1=k_1\int_\delta^\eta d\eta\int_\delta^\eta\frac{\sin\{\sqrt{\tfrac{3}{2}}\ \text{Re}\,[\tfrac{1}{2}\eta(1-\eta^2)^{\frac{1}{2}}+\tfrac{1}{2}\sin^{-1}\eta]\}}{(1-\eta^2)^{\frac{1}{4}}}d\eta+k_8\,\eta.$$

In the second of the above equations vanishing of v_1 at $\eta=0$ has been utilized and K_8 is a constant to be determined by matching.

Near $\eta=1$, we let $z=(3\ Re)^{\frac{1}{3}}(1-\eta)$ in equation (13) to obtain the approximation

$$\frac{d^4 v_1}{dz^4}+z\frac{d^2 v_1}{dz^2}=0 \qquad \ldots\ (21)$$

This is second order equation in $\frac{d^2 v_1}{dz^2}$ with the solution given by

$$\frac{d^2 v_1}{az^2}=AA_i(-z)-BB_i(-z) \qquad \ldots\ (22a)$$

where A_i and B_i are Airy's functions

Integrating equation (22a), we get

$$\frac{dv_1}{dz}=A\int_0^z A_i(-z)dz+B\int_0^z B_i(-z)dz+C \tag{22b}$$

and

$$v_i=A\int_0^z dz\int_0^z A_i(-z)dz+B\int_0^z dz\int_0^z B_i(-z)dz+Cz+D. \tag{22c}$$

For small z (22b) and (22c) can be expressed as

$$\frac{dv_1}{dz}=-C_1(A+B\sqrt{3})\,F(-z)+C_2(A-B\sqrt{3})G(-z)+C$$

and

$$v_1=-4(A+B\sqrt{3})\int_0^z F(-z)dz+C_2(A-B\sqrt{3})\int_0^z G(-z)dz+Cz+D,$$

where

$$F(z)=\sum_0^\infty 3^k(\tfrac{1}{3})_k\frac{z^{3k+1}}{(3k+1)},$$

$$G(z)=3^k(\tfrac{2}{3})k\frac{z^{3k+2}}{(3k+2)},$$

$$C_1=A_i(0)=3^{-2/3}/\Gamma(2/3),$$

$$C_2=-A'_i(0)=3^{-1/3}/\Gamma(1/3),$$

Now applying the conditions,

$$v_1=-1 \text{ and } \frac{dv_1}{dz}=0 \text{ at } z=0$$

we get $C=0$ and $D=-1$.

The constants K_1, K_3, A and B are now to be determined by matching the solutions as represented by equations (20) and (22)

When expressed in terms of $z=(3\text{ Re})^{1/3}(1-\eta)$, and approximated for large Re, equation (20*a*) gives

$$\frac{d^2v_1}{dz^2}\sim\frac{k_1[\sin(\sqrt{\tfrac{3}{2}}\text{ Re}\tfrac{\pi}{4})\cos\tfrac{2}{3}z^{\frac{3}{2}}-\cos(\sqrt{\tfrac{3}{2}}\text{ Re}\tfrac{\pi}{4})\sin\tfrac{2}{3}z^{\frac{3}{2}}]}{(3\text{ Re})^{\frac{7}{12}}\,2^{\frac{1}{4}}\,z^{\frac{1}{4}}} \tag{24a}$$

Again for η not too near 1 and Re large, i.e.. for large z, we have the following asymptotic approximations of (22a, b, c).

$$\frac{d^2v_1}{dz^2} \sim \frac{1}{\sqrt{\pi}\ z^{\frac{1}{4}}}\left[A \sin\left(\tfrac{\pi}{4}+\tfrac{2}{3}z^{\frac{3}{2}}\right)+B\cos\left(\tfrac{\pi}{4}+\tfrac{2}{3}z^{\frac{3}{2}}\right)\right] \quad \dots (25a)$$

$$\frac{dv_1}{dz} \sim A\left[\frac{2}{3}-\frac{1}{\sqrt{\pi}\ z^{\frac{3}{4}}}\cos\left(\tfrac{\pi}{4}+\tfrac{2}{3}z^{\frac{3}{2}}\right)+\right.$$

$$\left.+\frac{B}{\sqrt{\pi}\,z^{\frac{3}{4}}}\sin\left(\tfrac{\pi}{4}+\tfrac{2}{3}z^{\frac{3}{2}}\right),\right. \quad \dots (25b)$$

$$v_1 \sim A(C_2-\tfrac{2}{3}z)-BC_2\sqrt{3}-1$$

$$\sim A[C^2-\tfrac{2}{3}(3\ \text{Re})^{\frac{1}{3}}(1-\eta)]-BC_2\sqrt{3-1}. \quad \dots (25c)$$

Comparing the values of $\frac{d^2v_1}{dz^2}$ as given by (24a) and (25a) we have

$$A+B=\frac{K_1 \sin \sqrt{\frac{3}{2}}\,\text{Re}\ \frac{\pi}{4}}{(3\ \text{Re})^{\frac{7}{12}}\ 2^{\frac{1}{4}}}\sqrt{2\pi} \quad \dots (26a)$$

and

$$A-b=-\frac{K_1\cos\left(\sqrt{\frac{3}{2}}\,\text{Re}\ \frac{\pi}{4}\right)\sqrt{2\pi}}{(3\ \text{Re})^{\frac{7}{12}}\ 2^{\frac{1}{4}}} \quad \dots (26b)$$

Again matching v_1 and $\frac{dv_1}{dz}$ as given by (25c) and (25b) with the values expressed by (20c) and (20b) we have

$$A\left[C_2-\tfrac{2}{3}(3\ \text{Re})^{\frac{1}{3}}\right]-C_2\sqrt{3}\ B-1=0 \quad \dots (26c)$$

and

$$\tfrac{2}{3}(3\ \text{Re})^{\frac{1}{3}}A=K_3 \quad \dots (26d)$$

From the above four equations, viz., (26a, b, c, d) the four constants can be evaluated.

The centre section velocity is given by

$$\frac{u(0)}{q_0}=\frac{3}{2}+\frac{\xi}{\alpha}\left(\frac{dv_1}{d\eta}\right)_0=\frac{3}{2}+\frac{k_3}{\alpha}$$

$$=\frac{3}{2}-\frac{\xi}{\alpha\left[1+\dfrac{c^2\{\sqrt{3}\cot\frac{\pi}{4}(\sqrt{\frac{3}{2}}\,\mathrm{Re}-1)-1\}}{(\frac{2}{3}\,\mathrm{Re})^{\frac{1}{3}}}\right]}$$

Since Re is large in this case the above expression shows the decrease in the value of $u(0)$ as we move backward from the fully developed region.

REFERENCES

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