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THE GLOBAL DOMINATION NUMBER OF FUZZY GRAPHS

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Abstract

A Dominating set D of a fuzzy graph $G = (\mu, \rho)$ is a global dominating set (g.d.set) if D is also dominating set of the complement $\overline{a}$ of G. The global dominating number γ_g of G is the minimum fuzzy cardinality of a g.d.set. In this paper we introduce and study the concept of global dominating set of fuzzy graphs and investigate the relationshiop of $\gamma_g(G)$ with other known parameters of G. Moreover, we obtain many bounds on $\gamma_g(G)$ and we give the exact values of $\gamma_g(G)$ for some standard fuzzy graphs. Also, we give a Nordhaus - Gaggum type result for this parameter.

Finally, we present and study the concepts of a full number, a global full number of fuzzy graphs and a global domatic number of fuzzy graphs.

Keywords : Fuzzy graph, domination number, global domination number, domatic number, global domatic number, full and global full number.

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1. Introduction

Rosenfeld (1975) introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as path, cycles, connectedness and etc. The fuzzy relations introduced by Zadeh. Bhattacharya (1987) and Bhutani (1989) investigated the concept of fuzzy automorphism graphs. McAlester (1988) presented a generalization of intersection graphs to fuzzy intersection graphs. Mordeson (1993) introduced the concept of fuzzy line graphs and developd its basic properties. The concept of domination in fuzzy graphs was investigated by Somasundaram (1998). Fuzzy neighborhood set, fuzzy covering and split dominating number of fuzzy graphs was investigated by Mahioub and Soner (2007). In this paper we introduce the concept of global domination in fuzzy graphs.

2. Definition

We review briefly some definitions in fuzzy graphs. Notation and terminology not introduced here is found in [7,8]. Let V be a finite nonempty set. Let E be the collection of all two-element subset of V. A fuzzy graph $G = (\mu, \rho)$ is a set with two function $\mu : V \to [0,1]$ and $\rho : E \to [0,1]$ such that $\rho(\{x, y\}) \leq \mu(x) \wedge \mu(y)$ for all $x, y \in V$. We write $\rho(xy)$ for $\rho(\{x,y\})$. The order p and size q of a fuzzy graph $G = (\mu, \rho)$ are defined to be $p = \sum_{x \in V} \mu(x)$ and $q = \sum_{xy \in E} \rho(xy)$.

The complement of a fuzzy graph G denoted by $\overline{G}$ is defined to be $\overline{G} = (\mu, \overline{\rho})$ where $\overline{\rho}(xy) = \mu(x) \wedge \mu(y) - \rho(xy)$.

Let $G = (\mu, \rho)$ be a fuzzy graph on V. Let $u,v \in V$. We say that u dominates v in G if $\rho = (uv) = \mu(u) \wedge \mu(v)$. A subset D of V is called a dominating set in G if for every $v \in V - D$, there exists $u \in D$ such that u dominiates v. The minimum fuzzy cardinality of dominating sets in G is called the domination number of G and is denoted by $\gamma(G)$. A dominating sets D of fuzzy cardinality $\gamma(G)$ is called a minimum dominating set or γ-set. A dominating set D of a fuzzy graph G is said to be a minimal dominating set if no proper subset of S is a dominating set of G. The maximum fuzzy cardinality of minimal dominating sets is called the upper dominating number of G and is denoted by $\Gamma(G)$.

Let $G = (\mu,\rho)$ be a fuzzy graph on V. A subset D of V is said to be an independent set if $\rho(uv) < \mu(u) \wedge \mu(v)$ for all $u,v \in D$. The maximum fuzzy cardinality of independent sets in G is called the independence number of G and is denoted by $\beta(G)$. A dominating set D of a fuzzy graph (μ, ρ) is called connected dominating set of G if the induced fuzzy subgraph $\langle D \rangle$ is connected. The connected dominating number of a fuzzy graph G is the minimum fuzzy cardinality of connected dominating sets in G and is denoted by $\gamma_c(G)$. A dominating set D of G is called independent dominating set if D is independent. The independence domination number of G is the minimum fuzzy cardinality of the independent dominating sets in G and is denoted by $\gamma_i(G)$.

3. Main Results

Definition 3.1: Let G be a fuzzy graph. A dominating set D of G is a global dominating set (*g.d.* set) of G if D is also a dominating set of the complement $\overline{G}$ of G. The global dominating number of G is the minimum fuzzy cardinality of global dominating sets of G and is denoted by $\gamma_g(G)$ or simply γ_g.

Theorem 3.1: A dominating set D of a fuzzy graph G is a g.d. set if, and only if, for each $v \in V - D$, there exists a vertex $u \in D$ such that $\rho(u,v) = 0$.

Proof: Let D be a dominating set of a fuzzy graph G, D is a global dominating set if an only if for each $v \in V- D$ $\exists u \in D$ such that $\overline{\rho}(u, v) = \mu(u) \wedge \mu(v)$, (i.e) $\rho(u, v) = 0$.

Example 3.1: Let G be a fuzzy graph given in figure 3.1 with $\mu(v_1) = 0.2$, $\mu(v_2) = 0.3$, $\mu(v_3) = 0.4$, $\mu(v_4) = 0.5$, $\mu(v_5) = 0.6$, and $\rho(v_1, v_2) = 0.2$, $\rho(v_2, v_3) = 0.1$, $\rho(v_3, v_4) = 0.4$, $\rho(v_3, v_5) = 0.4$. Then $D = \{v_1, v_2, v_3\}$ is a global dominating set of G with $\gamma_g(G) = 0.9$.

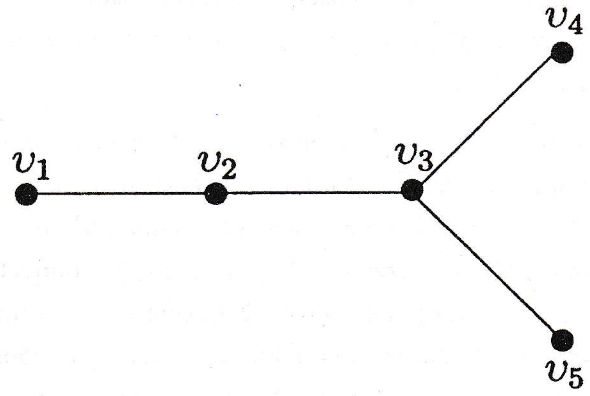

Fig. 3.1

The following theorem follows directly from the definitions.

Theorem 3.2: For any fuzzy graph G,

$$\gamma_g(G) = \gamma_g(\overline{G}) \tag{3.1}$$

$$\gamma(G) \leq \gamma_g(G) \tag{3.2}$$

$$\frac{\gamma + \overline{\gamma}}{2} \leq \gamma_g(G) \leq \gamma + \overline{\gamma} \tag{3.3}$$

Theorem 3.3: If G is a fuzzy graph such that G is independent, then

$$\gamma(G) = \gamma_g(G) = \Gamma(G) = p \tag{3.4}$$

If G is not independent and $\overline{G}$ is independent, then $\gamma(G)$ need not be equal to $\gamma_g(G)$. For example. But $\gamma_g(K_\mu) = p$.

Theorem 3.4: For any fuzzy graph G, if $\overline{G}$ is independent, then

$$\gamma_g(G) \geq \beta(G) \tag{3.5}$$

Equality hold if G is independent.

Proof: Let D be a global dominating set of G. Let S be an independent set of G, if $\overline{G}$ is independent, then $\gamma_g(G) = p \geq \beta(G)$. If G is also indpendent, then $\gamma_g(G) = \beta(G) = p$.

For the global domination number $\gamma_g(G)$ the following theorem gives a Nordhaus - Gaddum type result.

Theorem 3.5: For any fuzzy graph G,

$$\gamma_g(G) + \gamma_g(\overline{G}) \leq 2p \tag{3.6}$$

and the equality holds if $\rho(u,v) < \mu(u) \wedge \mu(v), \forall u, v \in V$.

Proof: Since V it self a global dominating set of G, then $\gamma_g(G) \leq |V| = p$ and $\gamma_g(\overline{G}) \leq |V| = p$. Hence $\gamma_g(G) + \gamma_g(\overline{G}) \leq 2p$.

If $\rho(u,v) < \mu(u) \wedge \mu(v), \forall u, v \in V$, then $\gamma_g(G) = \gamma_g(\overline{G}) = p$. Hence $\gamma_g(G) + \gamma_g(\overline{G}) = 2p$.

Proposition 3.1: Let D be a minimum dominating set of G. If there exists a vertex v in V - D adjacent to only vertices in D, then

$$\gamma_g(G) \leq \gamma + \max\{\mu(v) : v \in V - D\} \tag{3.7}$$

Proof: This follows, since $D \cup \{v\}$ is a global dominating set.

Corollary 3.1: Let $G = (K_{\mu_1,\mu_2})$ be a bipartite fuzzy graph without isolated vertex, where $|\mu_1| = \sum_{u \in V_1} \mu(v) = m, |\mu_2| = \sum_{u \in V_2} \mu(u) = n$ and $m \leq n$. Then

$$\gamma_g(G) \leq m + \max\{\mu(v) : v \in V - D\} \tag{3.8}$$

Corollary 3.2: If D is a minimum dominating set of G and V - D is independent, then

$$\gamma_g(G) \leq m + \max\{\mu(v) : v \in V - D\} \tag{3.9}$$

Let α and β respectively, denote the covering and independence numbers of a fuzzy graph G.

Theorem 3.6: Let G be a fuzzy graph. If for each $(u,v) \in \rho^*$, $\rho(u,v) = \mu(u) \wedge \mu(v)$, then

$$\gamma_g(G) \leq p - \beta(G) + 1 \quad (3.10)$$

Proof: Let I be an independent set with $|I| = \beta$. Since G has no isolated vertex, $V - I$ is a dominating set of G. Clearly for any vertex $v \in I, \{V - I\} \cup \{v\}$ is a global dominating set of G, and the upper bound follows.

Since $\alpha + \beta = p$ for any fuzzy graph G of order p without isolated vertex, we have the following result from (3.10),

Corollary 3.3:

$$\gamma_g(G) \leq \alpha(G) + 1 \quad (3.11)$$

Proposition 3.2: For any fuzzy graph G,

$$\gamma_g(G) \leq \gamma_i(G) \leq \beta(G) \quad (3.12)$$

and the equality holds if $\rho(u,v) < \mu(u) \wedge \mu(v), \forall u, v \in V$.

Proof : Let D be an independent dominating set of G, then D is a dominating set of G. Hence $\gamma(G) \leq \gamma_i(G)$. And since D is independent. Hence $\gamma_i(G) \leq \beta$. If $\rho(u,v) < \mu(u) \wedge \mu(v), \forall u, v \in V$. Then $\gamma = \gamma_i = \beta$.

As a consequence of (3.10) and (3.12), we have

Corollary 3.4: For any fuzzy graph G of order p without isolates.

$$\gamma + \gamma_g \leq p + 1 \quad (3.13)$$

$$\gamma_i + \gamma_g \leq p + 1 \quad (3.14)$$

The following proposition gives the relationship between $\gamma_c(G)$ and $\gamma_g(G)$ in fuzzy graphs.

Proposition 3.3: For any fuzzy graph G, at least one of the following holds:

$$\gamma_c(G) \le \gamma_g(G) \tag{3.15}$$

$$\gamma_c(\overline{G}) \le \gamma_g(G) \tag{3.16}$$

Proof: Let D be γ_g - set of G. Then D induce a connected fuzzy subgraph in G or $\overline{G}$. Hence D is a connected dominating set of G or $\overline{G}$. Thus, (3.15) and (3.16) hold.

Now a consequence of (3.11), we get

Corollary 3.5: For any fuzzy graph G, at least one of the following holds.

$$\gamma_c(G) \le \alpha(G) + 1 \tag{3.17}$$

$$\gamma_c(\overline{G}) \le \alpha(G) + 1 \tag{3.18}$$

Definition 3.2: Let G be a fuzzy graph on V. A subset S of V is full if $N(v) \cap (V - S) \ne \phi \ \forall v \in S$. The full number of a fuzzy graph G is the maximum fuzzy cardinality of full sets of G and denoted by $f(G)$.

Definition 3.3: Let G be a fuzzy graph on V. A subset S of V is g-full if $N(v) \cap (V - S) \ne \phi \ \forall v \in S$ both in G and $\overline{G}$. The g-full number of a fuzzy graph G is the maximum fuzzy cardinality of g-full sets of G and denoted by $f_g(G)$.

Observation 3.1: For any fuzzy graph G,

$$f_g(G) = f_g(\overline{G}) \tag{3.19}$$

Theorem 3.7: If G is a fuzzy graph of order p such that G has at least one effective edge, then

$$\gamma(G) + f(G) = p \tag{3.20}$$

Proof: Let D be a γ - set of G and $v \in V - D$. Then $N(v) \cap D \neq \phi$ in G. Hence, $V - D$ is a full and $p - \gamma \leq f$. On the otherhand. Suppose $S \subseteq V$ is a full with $|S| = f$. Then, for all $v \in S$, $N(v) \cap (V - S) \neq \phi$ in G. This implies that $V - S$ is a dominating set. Hence $\gamma \leq |V - S| = p - f$ and (3.20) holds.

Analogously we have.

Theorem 3.8: If G is a fuzzy graph of order p such that G has at least one effective edge, then

$$\gamma_g(G) + f_g(G) = p \tag{3.21}$$

Proof: Let D be a global dominating set of G with a minimum fuzzy cardinality and $v \in V - D$. Then $N(v) \cap D \neq \phi$ both in G and $\overline{G}$. Hence $V - D$ is a g-full and $p - \gamma_g = |V - D| \leq f_g$. On the otherhand. Suppose $S \subseteq V$ is a g-full with $|S| = f_g$. Then, for all $v \in S$, $N(v) \cap (V - S) \neq \phi$ both in G and $\overline{G}$. This implies that $V - S$ is a global dominating set. Hence $\gamma_g \leq |V - S| = p - f_g$ and (3.21) holds.

Corollary 3.6: If $F \subseteq V$ is a g-full, then

$$\gamma_g(G) \leq p - |F| \tag{3.22}$$

Corollary 3.7: If G has no isolates, and $F \subseteq V$ such that $N(v) \cap N(u) = \phi$ for all $u, v \in F$, then F is g-full, and $\gamma_g(G) \leq p - |F|$.

Corollary 3.8: If $G = K_\mu$ or $\overline{K}_\mu$, then $f_g = 0$. Since $\gamma \leq \gamma_g, \gamma(G) + f(G) = p$ and $\gamma_g(G) + f_g(G) = p$ we have

Corollary 3.9: for any fuzzy graph G,

$$f_g(G) \leq f(G) \tag{3.23}$$

Definition 3.4: Let G be a fuzzy graph on V. The private neighbor of a vertex $v \in V$ with respect to a set S, denoted by $PN\,[v, S]$, is the set $N[v]$ - $N\,[S\text{-}\{v\}]$. (i.e. $PN[v,S] = N[v] - N[S-\{v\}]$). If $PN[v, S] \neq \phi$ for some vertex v and some $S \subseteq V$, then every vertex of $PN[v, S]$ is called a private neighbor of v with respect to S.

Definition 3.5: A nonempty set $S \subseteq V$ is called in irredundant set of a fuzzy graph G if for each vertex $v \in S$, $N[v] - N[S-\{v\}] \neq \phi$. The irredundance number of a fuzzy graph G is the minimum fuzzy cardinality of irredundant sets and is denoted by $ir(G)$. The upper irredundance number of a fuzzy graph G is the maximum fuzzy cardinality of irredundant sets of G and is denoted by $IR(G)$.

Definition 3.6: An irredundant set S of a fuzzy graph G is a global irredundant (g-irredundant) set of G or ($\overline{G}$) if it is irredundant both in G and $\overline{G}$. The global irredundance number of a fuzzy graph G, is the minimum fuzzy cardinality of g-irredundance sets and is denoted by $ir_g(G)$.

Definition 3.7: A global irredundant set S of a fuzzy graph G is a maximal global irredundant set if there is no a global irredundant set S' with $|S'| > |S|$. The Upper global irredundance number of a fuzzy graph G is the maximum fuzzy cardinality of maximal global irredundant sets and is denoted by $IR_g(G)$.

Observation 3.2:

(i) if G is a fuzzy graph without isolates, then any irredundant set of G is a full.

(ii) if both G and $\overline{G}$ have no isolates, then any g-irredundant set of G is a g-full.

As a consequence of (3.25) an (3.26), we have

Proposition 3.4: If G is a fuzzy graph of order p and $0 < \delta\,(G) \leq \Delta(G) < p$ - 1, then

$$\gamma \leq p - IR(G) \qquad (3.24)$$

$$\gamma_g \leq p - IR_g(G) \qquad (3.25)$$

Definition 3.8: Let G be a fuzzy graph. A partition of $V(G)$ in to a fuzzy dominating sets is a collection P of a fuzzy dominating sets of G with no common vertex such that the union of them is $V(G)$.

Definition 3.9: Let $G = (\mu, \rho)$ be a fuzzy graph and $P = \{D_1, D_2, D_3, \ldots, D_m\}$ be a partition of $V(G)$ in to dominating sets. We define norm of P to be $\sum_{i=1}^{m} \frac{\mu(D_i)}{|D_i|}$, where $\mu(D_i) = \sum_{x \in D_i} \mu(x)$, and is denoted by $\|P\|$.

Definition 3.10: We define the fuzzy domatic number of G by the maximum norm of a partition of $V(G)$ in to a dominating sets of fuzzy graph G and is denoted by $d(G)$.

Observation 3.3: We say P is a maximum partition if P is a partition with $\|P\| = d(G)$.

Definition 3.11: A partition $P = \{D_1, D_2, D_3, \ldots, D_m\}$ of $V(G)$ is a global domatic partition of G if D_i is a global dominating set of G. The global domatic number of G is the maximum fuzzy cardinality of $\|P\| = \sum_{i=1}^{m} \frac{\mu(D_i)}{|D_i|}$ and is denoted by $d_g(G)$.

Example 3.2: Let $G = (\mu, \rho)$ be a fuzzy graph on V, such that $V = \{v_1, v_2, v_3, v_4\}$, and $\mu(v_1) = 0.3$, $\mu(v_2) = 0.4$, $\mu(v_3) = 0.4$, $\mu(v_4) = 0.5$, $\rho(v_1, v_2) = 0.3$, $\rho(v_2, v_3) = 0.4$, $\rho(v_3, v_4) = 0.4$. There is Three partitions of $V(G)$ in to global dominating sets. The trivial partition $P_1 = \{\{v_1, v_2, v_3, v_4\}\}$, $P_2 = \{\{v_1, v_3\}, \{v_2, v_4\}\}$, $P_3 = \{\{v_2, v_3\}, \{v_1, v_4\}\}$ with $\|P_1\| = \frac{1.6}{4} = 0.4$, $\|P_2\| = \frac{0.3+0.4}{2} + \frac{0.4+0.5}{2} = 0.8$, $\|P_3\| = \frac{0.4+0.4}{2} + \frac{0.3+0.5}{2} = 0.8$.

Thus, $d_g(G) = 0.8$.

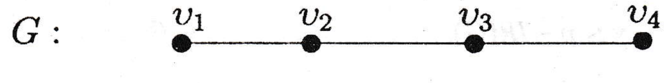

Fig. 3.2

Observation 3.4: For any Fuzzy graph G,

$$d_g(G) = d_g(\overline{G}) \tag{3.26}$$

Theorem 3.9: If G is a fuzzy graph of order p, then

$$d_g(G) \leq p \tag{3.27}$$

Proof: Suppose that $P = \{D_1, \ldots, D_m\}$ be a maximum partition we have $\sum_{i=1}^{m} \mu(D_i) = p$ and since any global dominating set is nonempty so $|D_i| \geq 1$. Hence the proof.

Observation 3.5: Since a global dominating set in a complete fuzzy graph G is V, then

$$d_g(K_\mu) = d_g(\overline{K}_\mu) = \frac{p}{n} : \text{and } d(K_\mu) = p \tag{3.28}$$

Example 3.3: Let $G = (\mu, \rho)$ be a fuzzy graph on V, given in the figure 3.3.

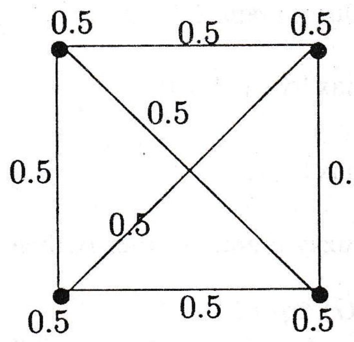

Fig. 3.3

As $D_1 = V$ is the only γ_g - *set*, $d_g(G) = \max\|P\| = \sum_{i=1}^{m} \frac{\mu(V)}{|V|} = \frac{2}{4} = \frac{2}{4} = 0.5$. But

$$d(G) = \frac{0.5}{1} + \frac{0.5}{1} + \frac{0.5}{1} + \frac{0.5}{1} = 2 = p.$$

Observation 3.6: Let $G = (K_{\mu_1,\mu_2})$ be a fuzzy bipartite fuzzy graph with order p. If $|V_1| = |V_2|$, then

$$d(G) = \frac{p}{2} \tag{3.29}$$

Proposition 3.5: For any fuzzy graph G of order p.

$$d_g(G) \leq d(G) \leq \delta(G) + 1 \tag{3.30}$$

$$d_g(G) \leq \frac{d + \overline{d}}{2} \tag{3.31}$$

$$d_g(G) \leq \min\{\delta + 1, \overline{\delta} + 1\} \tag{3.32}$$

Proof: The result in (3.30) follow from the definition. Also (3.30) and (3.26) imply (3.31), (3.32).

Corollary 3.10: For any fuzzy graph G

$$\gamma_g(G) \leq \max\{\Delta + 1, \overline{\Delta} + 1\} \tag{3.33}$$

$$\gamma_g(G) \leq \max\{\Delta, \overline{\Delta}\} \tag{3.34}$$

Theorem 3.10: If G is a fuzzy graph of order p, then

$$\gamma(G) + d(G) \leq p + 1 \tag{3.35}$$

$$\gamma_g(G) + d_g(G) \leq p + 1 \tag{3.36}$$

Proof: Clearly (3.40), (3.41) follows from (3.30), (3.32) and (3.33).

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