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FILTER CHARACTERIZATION ON FUZZY G_δ*-COMPACT SPACES

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Abstract

In this paper the concepts of fuzzy G_δ - compact space, fuzzy G_δ - compact* space and fuzzy G_δ - filters are introduced and studied. Some interesting properties and characterizations are established. Filter characterization on fuzzy G_δ - compact* space is also discussed.

Keywords : Fuzzy G_δ - compact space, fuzzy F_σ - perfect mapping, fuzzy G_δ - quasi perfect mapping, fuzzy G_δ - Lindelöf space, fuzzy G_δ- compact* space and fuzzy G_δ-filter.

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1. Introduction and Preliminaries

Ever since the introduction of fuzzy set by L.A. Zadeh [12], the fuzzy concept has invaded almost all branches of Mathematics. The concept of fuzzy topological space was introduced in [6] by C.L. Chang. Since then many fuzzy topologists [1,3,8 -11] have extended various notions in classical topology to fuzzy topological spaces. In this paper the concept of fuzzy G_δ-compact space is introduced based on the concept of fuzzy compact space by Bin Shahna [4] and fuzzy G_δ-sets by Balasubramanian [2]. Using the concept of covering property fuzzy G_δ-compact* space is introduced and studied. Some interesting properties and characterizations are also studied. The filter characterization on fuzzy G_δ-compact* space is discussed by making use of the concept of fuzzy G_δ-filters as in [5].

Definition 1.1: I denotes the unit interval [0,1] of the real line. Let X and Y be sets, α be an arbitary member of the index set A and B of R.

For X, I^X denotes the collection of mapping from X into I. A member λ of I^X is called a fuzzy set of X. The union $\vee \lambda_\alpha$ (intersection $\wedge \lambda_\alpha$) of a family $\{\lambda_\alpha\}$ of fuzzy sets of X is defined to be the mapping sup λ_α(inf λ_α). For any two members λ and μ of I^X, $\lambda \geq \mu$ iff $\lambda(x) \geq \mu(x)$ for each $x \in X$ and in this case λ is said to contain μ, or μ is said to be contained in λ. 0 and 1 denote constant mappings taking whole of X to 0 and 1 respectively.

Definition 1.2: Let λ and μ be any two fuzzy sets in the fuzzy topological space (X, T). Then we define $\lambda \vee \mu : X \rightarrow [0,1]$ as follows.

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\} \text{ for all } x \in X.$$

Also we define $\lambda \wedge \mu : X \rightarrow [0,1]$ as follows:

$(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$ for all $x \in X$ [6].

Definition 1.3: A fuzzy topology T on X is a collection of subsets of I^X such that

(a) $0, 1 \in T$,

(b) If $\lambda, \mu \in T$, then $\lambda \wedge \mu \in T$,

(c) If $\lambda_i \in T$ for each $i \in I$, then $\vee \lambda_i \in T$.

The ordered pair (X,T) is called a fuzzy topological space [6]. Every member of T is called a T-open fuzzy set. A fuzzy set is T-closed iff its complement is T-open.

Definition 1.4: Let $f: (X,T) \to (Y,U)$ be a mapping from a fuzzy topological space X to another fuzzy topological space Y. f is called

(a) a fuzzy continuous mapping [1] if $f^{1}(\lambda) \in T$ for each $\lambda \in S$ or equivalently $f^{-1}(\mu)$ is fuzzy closed set of X for each fuzzy closed set μ of Y.

(b) a fuzzy open (fuzzy closed) mapping [1] if $f(\lambda)$ is fuzzy open (fuzzy closed) set of X for each fuzzy open (fuzzy closed) set λ of X.

Lemma 1.1: Let $f: X \to Y$ be a mapping and $\{\lambda_\alpha\}$ be a family of fuzzy sets of Y, then

(a) $f^{-1}(\vee\lambda_\alpha) = \vee f^{-1}(\lambda_\alpha)$

and

(b) $f^{-1}(\wedge\lambda_\alpha) = \wedge f^{-1}(\lambda_\alpha)$ [1].

Notation 1.1: $1-\lambda$ denotes the complement of any fuzzy set λ.

Definition 1.5: Let (X,T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy G_δ - set [2] if $\lambda = \overset{\infty}{\underset{i=1}{\wedge}} \lambda_i$ where each $\lambda_i \in T$.

Definition 1.6: Let (X,T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy F_σ - set if $\lambda = \overset{\infty}{\underset{i=1}{\vee}} \lambda_i$ where each $1-\lambda_i \in T$.

Definition 1.7: A fuzzy filter F on the fuzzy topological space (X,T) is a non-empty collection of subsets of I^X with the following properties.

(i) $\lambda \in F$ is a fuzzy open-set in X

(ii) $0 \notin F$

(iii) $\mu_1, \mu_2 \in F$ then $\mu_1 \wedge \mu_2 \in F$

(iv) If $\mu \in F$ and γ is a fuzzy open-set in X with $\mu \leq \gamma$, then $\gamma \in F$ [5].

Definition 1.8: A fuzzy filter F on a fuzzy topological space (X,T) is fuzzy ultra filter if there is no other fuzzy filter finer than F[5].

Definition 1.9: A fuzzy filter F on a fuzzy topological space (X,T) is a fuzzy prime filter, if γ and μ are two fuzzy closed sets such that $\gamma \vee \mu \in F$ then $\gamma \in F$ or $\mu \in F$ [5].

2. Fuzzy G_δ - Compact Space

In this section the concept of fuzzy G_δ-compact space and fuzzy G_δ-compact* space are introduced. Some interesting properties and characterization are studied.

S denotes the fuzzy space consisting of a single point with the fuzzy topology $\{0,1\}$ [4]. Friedler [7] shows that if (X,T) is a fuzzy topological space, then $S \times X$ is fuzzy homeomorphic to X [4].

Note 2.1: Let (X,T) be a fuzzy topological space. For any fuzzy topological space (Z,R), p_{2X} denotes the projection of $X \times Z$ to Z.

Definition 2.1: Let f: $(X, T) \rightarrow (Y, U)$ be a mapping from a fuzzy topological space (X,T) to another fuzzy topological space (Y,U).

1. f is said to be fuzzy G_δ, if $f(\lambda)$ is a fuzzy G_δ-set in (Y, U) for any fuzzy G_δ-set λ in (X, T).

2. f is said to be fuzzy F_σ, if $f(\lambda)$ is a fuzzy F_σ-set in (Y, U) for any fuzzy F_σ-set λ in (X, T).

3. f is said to be M-fuzzy G_δ-continuous, if $f^{-1}(\lambda)$ is a fuzzy G_δ-set in (X,T) for any fuzzy G_δ-set λ in (Y, U).

Definition 2.2: A fuzzy topological space X is called fuzzy G_δ-space if every fuzzy G_δ-set of X is fuzzy open [4].

In 1979 Raha [8] studied the concepts of compact space, Lindelof space and G_δ-space. By using the concept of compact space [8], Bin Shahna [4] defined fuzzy compactness as follows. A fuzzy topological space X is said to be fuzzy compact if the projection $p_{2X} : X \times Z \to Z$ is fuzzy closed for any fuzzy topological space Z. Motivated by this concept fuzzy G_δ-compact space is introduced.

Definition 2.3: A fuzzy topological space (X, T) is said to be **fuzzy G_δ-compact** if the projection $p_{2X} : X \times Z \to Z$ is fuzzy F_σ for any fuzzy topological space (Z, R).

Proposition 2.1: An M-fuzzy G_δ-continuous image of a fuzzy G_δ-compact space is fuzzy G_δ-compact.

Proof: Let $f: (X, T) \to (Y, U)$ be a M-fuzzy G_δ-continuous mapping from a fuzzy G_δ-compact space (X, T) onto a fuzzy topological space (Y, U) and I_Z, be the identity mapping on a fuzzy topological space (Z, R). Then $f \times I_Z : X \times Z \to Y \times Z$ is M-fuzzy G_δ-continuous. Let μ be a fuzzy F_σ-set of $Y \times Z$. Then $(f \times I_Z)^{-1}(\mu)$ is fuzzy F_σ set of $X \times Z$. Since $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. $p_{2X}((f \times I_Z)^{-1}(\mu)) = p_{2Y}(\mu)$ is a fuzzy F_σ set of (Z,R) showing that p_{2Y} is a fuzzy F_σ - mapping. Hence (Y, U) is fuzzy G_δ-compact.

Definition 2.4: Let (X,T) and (Y,U) be any two fuzzy topological spaces. A fuzzy continuous map $f: (X, T) \to (Y, U)$ is fuzzy F_σ - perfect if $f \times I_Z : X \times Z \to Y \times Z$ is a fuzzy F_σ -map for any fuzzy topological space (Z, R).

Definition 2.5: An M-fuzzy G_δ-continuous mapping $f: (X,T) \to (Y,U)$ of a fuzzy topological space (X,T) into a fuzzy topological space (Y, U) is called a fuzzy G_δ-quasi perfect mapping if $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy F_σ for any fuzzy G_δ-space (Z,R).

Definition 2.6: A fuzzy topological space (X,T) is called fuzzy G_δ-Lindelöf if the projection $p_{2X} : X \times Z \to Z$ is a fuzzy F_σ -mapping for any fuzzy G_δ-space (Z,R).

Proposition 2.2: If $f: (X, T) \to (Y, U)$ is a fuzzy F_σ - perfect mapping of a fuzzy topological space (X,T) onto a fuzzy G_δ-compact space (Y,U) then (X,T) is fuzzy G_δ-compact.

Proof: Since f is fuzzy G_δ-perfect, $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy F_σ - mapping for any fuzzy topological space (Z,R). For (X,T) to be fuzzy G_δ-compact we show that $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. Noting that p_{2X} is the composition of two fuzzy F_σ - mappings $f \times I_Z$ and $p_{2Y} : Y \times Z \to Z$, the result follows.

Proposition 2.3: Let (X,T) be a fuzzy topological space and (Y, U) be a fuzzy G_δ-compact space. Then the projection $p : X \times Y \to X$ is fuzzy F_σ - perfect mapping.

Proof: We show that $p \times I_Z : (X \times Y) \times Z \to X \times Z$ is a fuzzy F_σ - mapping for any fuzzy topological space (Z,R). Since (Y,U) is fuzzy G_δ -compact, $p_{2Y} : Y \times (X \times Z) \to (X \times Z)$ is fuzzy F_σ . Then $p \times I_Z$ is fuzzy F_σ follows by noting that it is the composition $p_{2Y} \circ h$, where $h : (X \times Y) \times Z \to Y \times (X \times Z)$ is a fuzzy G_δ-homeomorphism.

Proposition 2.4: A fuzzy topological space (X, T) is fuzzy G_δ-compact if and only if the constant mapping $c : X \to S$ is fuzzy F_σ - perfect.

Proof: Suppose (X, T) is fuzzy G_δ-compact. We show that $c \times I_Z : X \times Z \to S \times Z$ is fuzzy F_σ for any fuzzy topological space (Z, R). Since (X, T) is fuzzy G_δ-compact $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. Note that $S \times Z$ is fuzzy G_δ-homeomorphic to (Z, R). Now $c \times I_Z = h \circ p_{2X}$, being a composition of a fuzzy F_σ - mapping p_{2X} and a fuzzy G_δ -homeomorphism $h : Z \to S \times Z$ is fuzzy F_σ. Hence c is fuzzy F_σ -perfect.

Conversely, if $c : X \to S$ is fuzzy G_δ -perfect, then $c \times I_Z : X \times Z \to S \times Z$ is fuzzy F_σ for any fuzzy topological space (Z, R). We show that $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. Since $p_{2X} = h \circ (c \times I_Z)$, where $h : S \times Z \to Z$ is a fuzzy G_δ-homeomorphism, p_{2X} is fuzzy F_σ and hence (X,T) is fuzzy G_δ-compact.

Proposition 2.5: The product of two fuzzy G_δ-compact spaces is fuzzy G_δ-compact.

Proof: Let (X,T) and (Y,U) be any two fuzzy G_δ -compact spaces. Then $p_{2X} : X \times (Y \times Z) \to (Y \times Z)$ and $p_{2Y} : Y \times Z \to Z$ are fuzzy F_σ - mappings for any fuzzy topological space (Z,R). We show that $p_{2(X \times Y)} : (X \times Y) \times Z \to Z$ is fuzzy F_σ. Since $p_{2(X \times Y)} = p_{2Y} \circ p_{2X}$ $\{X \times (Y \times Z)$ and $(X \times Y) \times Z$ are fuzzy homeomorphic$\}$, being a composition of two fuzzy F_σ - mappings, is fuzzy F_σ and hence $X \times Y$ is fuzzy G_δ-compact.

Note 2.2: The finite product of fuzzy G_δ-compact spaces is fuzzy G_δ-compact.

Proposition 2.6: Let (X_i, T_i), $i = 1, 2, 3$ be any three fuzzy topological spaces. The composition $g \circ f : (X_1, T_1) \to (X_3, T_3)$ of fuzzy G_δ -quasi perfect mappings $f : (X_1, T_1) \to (X_2, T_2)$ and $g : (X_2, T_2) \to (X_3, T_3)$ is fuzzy G_δ -quasi perfect.

Proof: Let (Z, R) be a fuzzy G_δ -space. Then the mapping $(g \circ f) \times I_Z : X_1 \times Z \to X_3 \times Z$ is fuzzy F_σ follows from the identify $(g \circ f) \times I_Z = (g \times I_Z) \circ (f \times I_Z)$ by noting that f and g are fuzzy G_δ-quasi perfect mappings.

Proposition 2.7: Let (X_i, T_i), $i = 1, 2, 3$ be any three fuzzy topological spaces.

Let $f : (X_1, T_1) \to (X_2, T_2)$ and $g : (X_2, T_2) \to (X_3, T_3)$ be M-fuzzy G_δ-continuous mappings. Then

(a) if $g \circ f$ is fuzzy G_δ-quasi perfect and f is surjective, then g is fuzzy G_δ- quasi perfect.

(b) if $g \circ f$ is fuzzy G_δ-quasi perfect and g is injective, then f is fuzzy G_δ-quasi perfect.

Proof: (a) We show that $g \times I_Z : X_2 \times Z \to X_3 \times Z$ is fuzzy F_σ - mapping for any fuzzy G_δ -space (Z, R). Let μ be a fuzzy F_σ - set of $X_2 \times Z$. Then $(f \times I_Z)^{-1}(\mu)$ is a fuzzy F_σ-set of $X_1 \times Z$. Because $(g \circ f) \times I_Z$ is fuzzy F_σ and $((g \circ f) \times I_Z)\,(f \times I_Z)^{-1}\,(\mu) = (g \times I_Z)\,(\mu)$, it follows that $(g \times I_Z)\,(\mu)$ is fuzzy F_σ-set of $X_3 \times Z$.

(b) We show that $f \times I_Z : X_1 \times Z \to X_2 \times Z$ is fuzzy F_σ - mapping for any fuzzy G_δ-space (Z, R). Let μ be a fuzzy F_σ - set of $X_1 \times Z$. Then $((g \circ f) \times I_Z)\,(\mu)$ is a fuzzy F_σ - set of $X_3 \times Z$. Because $g \times I_Z$ is M-fuzzy G_δ-continuous and $(g \times I_Z)^{-1}\,((g \circ f) \times I_Z)\,(\mu) = (f \times I_Z)\,(\mu)$, it follows that $(f \times I_Z)\,(\mu)$ is fuzzy F_σ - set of $X_2 \times Z$.

Proposition 2.8: An M-fuzzy G_δ -continuous image of a fuzzy G_δ-Lindelöf space is fuzzy G_δ -Lindelöf.

Proof: Let f be a M-fuzzy G_δ-continuous mapping from a fuzzy G_δ-Lindelöf space (X, T) onto a fuzzy G_δ-Lindelöf space (Y, U), and I_Z, be the identity mapping on a fuzzy topological space (Z, R). Then $f \times I_Z : X \times Z \to Y \times Z$ is M-fuzzy G_δ-continuous. Let μ be a fuzzy F_σ - set of $Y \times Z$. Then $(f \times I_Z)^{-1}\,(\mu)$ is a fuzzy F_σ - set of $X \times Z$. Since $p_{2X} : X \times Z \to Z$ is fuzzy F_σ - mapping $p_{2X}((f \times I_Z)^{-1})\,(\mu) = p_{2Y}\,(\mu)$ is a fuzzy F_σ - set of (Z, R) showing that P_{2Y} is fuzzy F_σ - mapping. Hence (Y, U) is fuzzy G_δ-Lindelöf.

Proposition 2.9: If $f : (X, T) \to (Y, U)$ is a fuzzy G_δ -quasi perfect mapping of a fuzzy topological space (X, T) onto a fuzzy G_δ -Lindelöf space (Y, U), then (X, T) is fuzzy G_δ -Lindelöf.

Proof: Since f is fuzzy G_δ -quasi perfect mapping, $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy F_σ for any fuzzy G_δ -space (Z, R). For (X, T) to be fuzzy G_δ-Lindelöf we show that $p_{2X} : X \times Z \to Z$ is fuzzy F_σ . Noting that p_{2X} is the composition of two fuzzy F_σ -mappings $f \times I_Z$ and $p_{2Y} : Y \times Z \to Z$, the result follows.

Proposition 2.10: Let (X, T) be a fuzzy G_δ -space and (Y, U) be a fuzzy G_δ -Lindelöf space. Then the projection $p : X \times Y \to X$ is a fuzzy G_δ -quasi perfect mapping.

Proof: We show that $p \times I_Z : (X \times Y) \times Z \to X \times Z$ is a fuzzy F_σ - mapping for any fuzzy G_δ-space (Z, R). Since (Y, U) is fuzzy G_δ -Lindelöf, $p_{2Y} : Y \times (X \times Z) \to (X \times Z)$ is fuzzy F_σ . That $p \times I_Z$ is fuzzy F_σ follows by noting that it is the composition $p_{2Y} \circ h$, where $h : (X \times Y) \times Z \to Y \times (X \times Z)$ is a fuzzy G_δ -homeomorphism.

Proposition 2.11: A fuzzy topological space (X,T) is fuzzy G_δ -Lindelöf iff the constant mapping $c : X \to S$ is fuzzy G_δ - quasi perfect.

Proof: Suppose (X, T) is fuzzy G_δ -Lindelöf. We show that $c \times I_Z : X \times Z \to S \times Z$ is fuzzy F_σ - mapping for any fuzzy G_δ -space (Z, R). Since (X, T) is fuzzy G_δ -Lindelöf, $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. Note that $S \times Z$ is fuzzy G_δ-homeomorphic to Z. Now $c \times I_Z = h \circ p_{2X}$, being the composition of fuzzy F_σ -mapping p_{2X} and a fuzzy G_δ-homeomorphism $h : Z \to S \times Z$ is fuzzy F_σ. Hence c is fuzzy G_δ-quasi perfect.

Conversely, if $c : X \to S$ is fuzzy G_δ -quasi perfect, then $c \times I_Z : X \times Z \to S \times Z$ is fuzzy F_σ for any fuzzy G_δ -space (Z, R). We show that $p_{2X} : X \times Z \to Z$ is fuzzy F_σ. Since $p_{2X} = h \circ (c \times I_Z)$, where $h : S \times Z \to Z$ is a fuzzy G_δ -homeomorphism, p_{2X} is fuzzy F_σ and hence (X, T) is fuzzy G_δ -Lindelöf.

Proposition 2.12: A product of two fuzzy G_δ -Lindelöf fuzzy G_δ -spaces is a fuzzy G_δ -Lindelöf G_δ -space.

Proof: Let (X, T) and (Y, U) be any two fuzzy G_δ -Lindelöf G_δ -spaces. Then $X \times Y$ is a fuzzy G_δ -space. It remains to show that $X \times Y$ is fuzzy G_δ -Lindelöf. Since (X, T) and (Y, U) are fuzzy G_δ -Lindelöf spaces, $p_{2X} : X \times (Y \times Z) \rightarrow Y \times Z$ and $p_{2Y} : Y \times Z \rightarrow Z$ are fuzzy F_σ - mappings. Clearly, $p_{2(X \times Y)} = p_{2Y} \circ p_{2X}$ (($X \times (Y \times Z)$ and $X \times (Y \times Z)$ are fuzzy homeomorphic), being a composition of two fuzzy F_σ - mappings, is fuzzy F_σ and hence $X \times Y$ is fuzzy G_δ-Lindelöf.

Proposition 2.13: Let (X, T) be a fuzzy G_δ-compact space. If (Y, U) is fuzzy G_δ -Lindelöf space, then $X \times Y$ is a fuzzy G_δ -Lindelöf.

Proof: It is similar to that of Proposition 2.5.

Using the concept of **covering property** we shall give the following definition.

Definition 2.7: A collection $\{\lambda_i\}_{i \in J}$ of fuzzy sets of fuzzy topological space (X, T) is called fuzzy G_δ -cover of (X,T) if λ_i's $(i \in J)$ are fuzzy G_δ -sets of (X, T). A fuzzy topological space (X, T) is called **fuzzy G_δ -compact*** if every fuzzy G_δ-cover of (X, T) has a finite subcover.

Example 2.1: Let n be any positive integer. Let (X_n, T_n) be a fuzzy topological space, where $T_n = \{0x_n, 1x_n, \lambda_n\}$. Define $\lambda_n : X_n \rightarrow [0,1]$ as $\lambda_n(x) = 1 - 1/n$ for all $x \in X_n$. Then (X_n, T_n) is fuzzy G_δ - compact* (countably fuzzy G_δ - compact*) sapce.

Proposition 2.14: The following are equivalent for a fuzzy topological space (X, T).

(a) (X, T) is fuzzy G_δ - compact*.

(b) For any family of fuzzy F_σ - sets $\{\lambda_i\}_{i \in J}$ with the property that $\wedge_{j \in F} \lambda_j \neq 0$ for any finite subset F of J, we have $\wedge_{i \in J} \lambda_i \neq 0$.

Proof (a) ⇒ (b)

Assume that (X, T) is fuzzy G_δ - compact*. Let $\{\lambda_i\}_{i \in J}$ be the family of fuzzy F_σ - set with the property that $\wedge_{j \in F} \lambda_j \neq 0$ for any finite subset F of J. Let $\mu_i = 1 - \lambda_i$. The collection $\{\mu_i\}_{i \in J}$ is a family of all fuzzy G_δ -sets in (X, T). If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are finite number of fuzzy F_σ-sets in $\{\lambda_i\}_{i \in J}$, then

$$\bigwedge_{i=1}^{n} \lambda_i = 1 - \bigvee_{i=1}^{n} \mu_i \qquad \ldots (2.1)$$

Similarly,

$$\bigwedge_{i \in J} \lambda_i = 1 - \bigvee_{i \in J} \mu_i \qquad \ldots (2.2)$$

By hypothesis, $\bigwedge_{i=1}^{n} \lambda_i \neq 0$. Therefore, $\bigvee_{i=1}^{n} \mu_i \neq 1$. Since (X, T) is fuzzy G_δ -compact $\bigvee_{i=J} \mu_i \neq 1$. From (2.2) it follows that $\bigwedge_{i=J} \lambda_i \neq 0$.

(b) ⇒ (a).

Let us assume that (X,T) is not fuzzy G_δ-compact. Let $\{\mu_i\}_{i \in J}$ be a family of fuzzy G_δ-sets which is a cover for (X,T). Since (X,T) is not fuzzy G_δ-compact, there is no finite subset F of J such that $\{\mu_j\}_{j \in F}$ is a cover for (X,T). i.e., $\bigvee_{j \in F} \mu_j \neq 1$. From this it follows that $\bigwedge_{j \in F} (1-\mu_j) = 1 - \bigvee_{i \in F} \mu_j \neq 0$. By hypothesis, $\bigwedge_{i=J} (1-\mu_i) \neq 0$. This implies that $\bigvee_{i \in J} \mu_i \neq 1$. Contradiction. Hence (X,T) is fuzzy G_δ-compact*.

Proposition 2.15: Let $f : (X, T) \rightarrow (Y, U)$ be an M-fuzzy G_δ-continuous surjective function of a fuzzy G_δ-compact* space (X, T) onto a fuzzy topological space (Y, U). Then (Y, U) is fuzzy G_δ-compact*.

Proof: Let $\{\lambda_i\}_{i \in J}$ be a collection of fuzzy G_δ sets of (Y, U) such that

$$1_Y \leq \bigvee_{i \in J} \lambda_i \qquad (2.3)$$

Since f is M-fuzzy G_δ-continuous and each λ_i is fuzzy G_δ in (Y, U), $f^{-1}(\lambda_i)$ is fuzzy G_δ in (X, T). From (2.3) $1x = f^{-1}(1_Y) \leq \bigvee_{i \in J} f^{-1}(\lambda_i)$. That is, $\{f^{-1}(\lambda_i)\}_{i \in J}$ is a fuzzy G_δ cover of (X, T). Since (X, T) is fuzzy G_δ-compact*, there exists a finite subset F of J such that

$$1_X \leq \bigvee_{i \in F} f^{-1}(\lambda_i) \Rightarrow f(1_X) = 1_Y \leq \bigvee_{i \in F} \lambda_i.$$

Therefore, (Y, U) is fuzzy G_δ-compact*.

Proposition 2.16: Let $f : (X, T) \rightarrow (Y, U)$ be a fuzzy G_δ, bijective function and (Y, U) be a fuzzy G_δ-compact* space. Then (X, T) is fuzzy G_δ-compact*.

Proof: Let $\{\lambda_i\}_{i \in J}$ be a collection of fuzzy G_δ-sets of (X, T) such that,

$$1_X \leq \bigvee_{i \in J} \lambda_i \qquad (2.4)$$

Since f is bijective $f(1_X) = 1_Y \leq f(\vee_{i\in J} \lambda_i) = \vee_{i\in J} f(\lambda_i)$. Since λ_i is fuzzy G_δ in (X, T) and f is fuzzy G_δ, $f(\lambda_i)$ is fuzzy G_δ in (Y, U). That is, $\{f(\lambda_i)\}_{i\in J}$ is a fuzzy G_δ -cover of (Y, U). Since (Y, U) is fuzzy G_δ -comapct*, there exists a finite subset F of J such that $1_Y \leq \vee_{i\in F} f(\lambda_i)$, $1x = f^{-1}(1_Y) \leq f^{-1}(\vee_{i\in F} f(\lambda_i)) = \vee_{i\in F} (\lambda_i)$. Therefore (X, T) is fuzzy G_δ -compact*.

The following **example** shows that the **Tychonoff product theorem for fuzzy G_δ-compact* (countably fuzzy G_δ-compact*) space is not valid**.

Example 2.2: Let n be any positive integer. Let (X_n, T_n) be a fuzzy topological space, where $T_n = \{0x_n, 1x_n, \lambda_n\}$. Define $\lambda_n : X_n \to [0, 1]$ as $\lambda_n(x) = 1 - 1/n$ for all $x \in X_n$. Then (X_n, T_n) is fuzzy G_δ-compact* (countably fuzzy G_δ-compact*) space. Let p_n be the projection of the product fuzzy topological space $X = \pi_{n=1}^{\infty} X_n$ onto X_n. Then $p_n^{-1}(\lambda_n(x)) = 1 - 1/n$ for $x \in X$. By the definition of the product fuzzy topological space the family $T_1 = \{0x, 1x, p_n^{-1}(\lambda_n), n = 1, 2 \ldots\}$ is used to generate the product fuzzy topology T by taking the finite intersection and then the arbitrary union of these intersections. Clearly, the product fuzzy topology generated is exactly T_1 itself. The family $\{p_n^{-1}(\lambda_n)\}$ $n = 1, 2 \ldots$ is a fuzzy G_δ -cover (countably fuzzy G_δ -cover) of (X, T) which has no finite subcover.

3. Fuzzy Filter Characterization on Fuzzy G_δ -Compact* Space

In this section fuzzy G_δ -filters are introduced. Filter characterization on fuzzy G_δ -compact* space is studied by making use of fuzzy G_δ -filters as in [5].

Definition 3.1: A fuzzy G_δ (F_σ)-filter F on the fuzzy topological space (X,T) is a non-empty collection of subsets of I^X with the following properties.

(i) $\lambda \in F$ is a fuzzy G_δ (F_σ)-set in X

(ii) $0 \notin F$

(iii) $\mu_1, \mu_2 \in F$ then $\mu_1 \wedge \mu_2 \in F$

(iv) If $\mu \in F$ and γ is a fuzzy G_δ (F_σ)-set in X with $\mu \leq \gamma$ then $\gamma \in F$.

Definition 3.2: A fuzzy G_δ (F_σ)-filter F on a fuzzy topological space (X, T) is fuzzy G_δ (F_σ)-ultra filter if there is no other fuzzy G_δ (F_σ)-filter finer than F.

Definition 3.3: A fuzzy F_σ-filter F on a fuzzy topological space (X, T) is a fuzzy F_σ-prime filter, if γ and μ are two fuzzy F_σ-sets such that $\gamma \vee \mu \in F$ then $\gamma \in F$ or $\mu \in F$.

Example 3.1: Define $X = \{a\}$. Let $T = \{0, 1, \lambda_n\}$ where $\lambda_n : X \to [0, 1]$ $(n = 2,3 \ldots, \infty)$ is such that $\lambda_n(a) = 1 - 1/n$, $n = 2,3, \ldots$. Therefore (X, T) is a fuzzy topological space.

Define $F = \{\lambda_n\}_{n=2}^{\infty}$. Clearly F is a fuzzy G_δ-filter and also a fuzzy G_δ-ultra filter.

Example 3.2: Define $X = \{a\}$. Let $T = \{\lambda_n\}$ where $\lambda_n : X \to [0,1]$ $(n = 2, 3, \ldots, \infty)$ is such that $\lambda_n(a) = 1/n$, $n = 1, 2, 3, \ldots$. Therefore (X, T) is a fuzzy topological space.

Define $F = \{1 - \lambda_n\}_{n=2}^{\infty}$. Clearly F is a fuzzy F_σ-filter and also a fuzzy F_σ-prime filter.

Proposition 3.1: The following are equivalent for a fuzzy topological space (X, T).

(a) (X, T) is fuzzy G_δ-compact*.

(b) Every fuzzy F_σ-filter F satisfies $\wedge_{\mu \in F} \mu \neq 0$.

(c) Every fuzzy F_σ-prime filter F satisfies $\wedge_{\mu \in F} \mu \neq 0$.

(d) Every fuzzy F_σ-ultra filter U satisfies $\wedge_{\mu \in U} \mu \neq 0$.

Proof (a) $\Rightarrow$ (b): Suppose $\wedge_{\mu \in F} \mu = 0$. Then, $\vee_{\mu \in F} (1-\mu) = 1$. Since $(1-\mu)$ is fuzzy G_δ and (X,T) is fuzzy G_δ-compact*, there must exist a finite sub collection $\{1-\mu_1, 1-\mu_2, \ldots 1-\mu_n\}$ such that $1 = (1 - \mu_1) \vee (1 - \mu_2) \vee \ldots (1 - \mu_n)$, that is $\mu_1 \wedge \mu_2 \wedge \ldots \wedge \mu_n = 0$ contradiction.

(b) ⇒ (c). Follows from the fact that every fuzzy F_σ-prime filter is fuzzy F_σ-filter.

(c) ⇒ (d). Obvious.

(d) ⇒ (a). Suppose H is a family of fuzzy F_σ-sets with finite intersection property. For each $\gamma \in H$ consider a family in the $G_\gamma = \{\mu : \mu \text{ is a fuzzy } F_\sigma\text{-set}, \mu \geq \gamma\}$.

Clearly, $\gamma \in G_\gamma$. Let $G = \bigcup_{\gamma \in H} \{G_\gamma\}$. Since H has the finite intersection property, G also has the property. Thus, there exists a fuzzy F_σ-ultra filter U such that $H \subset G \subset U$. Hence, $\wedge_{\mu \in H} \mu \geq \wedge_{\mu \in G} \mu \geq \wedge_{\mu \in U} \mu$. By hypothesis, $\wedge_{\mu \in U} \mu \neq 0$ and therefore $\wedge_{\mu \in H} \mu \neq 0$. This proves that (X, T) is fuzzy G_δ-compact*.

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