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MODELING THE INTERACTION OF SUSCEPTIBLE PREY WITH PREDATOR IN INFECTED ENVIRONMENT

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Abstract

In this paper, an interacting model involving susceptible prey with predator in infected environment has been proposed and analyzed. It is assumed that the infected prey which is susceptible produce infected environment by spreading its germs to the environment and the pred ator which is partially dependent on the prey but wholly dependent on the environment is affected. Criteria for local stability and global stability are established by constructing suitable Liapunov functions. Analyzing the stability condition it is found that if the rate of infection is not controlled in proper time then spread of an infectious disease may lead to a dramatic drop in prey as well as it affects the predator population with the entire ecosystem.

Key words : Susceptible prey, local stability, global stability.

1. Introduction

Biodiversity, the combination of species, genetic and ecological diversity (World Conservation Monitoring Centre (1992)) is one of the issues most frequently addressed in both scientific and mass media. Along with diversity, stability and complexity, resilience is a difficult ecological parameter to measure. It can be defined as the capacity to return to functional processes and interactions existing prior to disturbances (Pimm,1991; Shrader *et al.*,1993). For centuries, Scientists have been interested in the diversity of life forms and their evolution and extinction. One of the important problems in mathematical ecology is the interaction of biological species. The global dynamics of such interaction have been discussed by some researchers (Freedman ,1976; Bailey ,1975; Mishra *et al.*, 1993; . Fake *et al.*,2002; Hethcote et al., 2004;Das *et al.*, 2008). In particular, Freedman (1976) proposed a mathematical model for two dimensional predator prey interactions and examined the stability of the equilibrium. Liou and Cheng (1988) proposed a general model for predator-prey interaction and studied the global stability of the system. Sarkar and Ghosh (1997) studied a general model consisting of a micro-organism pool living on detritus of mangrove litter and its invertebrate predators. They found that the specific growth rate and the food conversion efficiency rate of the invertebrate predator plays a crucial role in governing the dynamics of such a system. A slight variation on the Lotka-Volterra predator-prey model gives rise to a new major field of study namely the modeling of infectious diseases. A great variety of infectious diseases inflict mankind but a few mathematical investigations have been made. Pokhariyal (1986) investigated the growth of infection using a mathematical model. Mishra *et al.* (1993) proposed a mathematical model to study the growth of infection taking into account the dynamics of parasite population growth and defense mechanism of the immune system. Flake *et al.* (2002) has studied a predator-prey model with diseases dynamics but with the infection of the susceptible prey only. Herbert W. Hethcote *et.al.* (2004) has studied a predator-prey model with infected prey. They have considered a parasitic infection in the prey. Das *et.al.* (2008) have studied the effect of diseases- selective predation on prey infected by contact and external sources but considering the predator completely avoids consuming the infected prey. Thus in this paper I have considered an ecosystem where the interaction between infectious diseases, susceptible prey and the predator in infected environment and have studied the stability criteria.

Mathematical Model

Let us consider an interacting model involving interaction between susceptible prey with predator in infected environment. It is also assumed that the system is governed by the Lotka-Volterra dynamics and is governed by the following autonomous differential equations:

$$\frac{dI}{dt} = I(a_{11} - a_{12}I + a_{13}x + a_{14}N)$$

$$\frac{dx}{dt} = x(a_{21} - a_{22}x - a_{23}N - a_{24}I) \quad \ldots (1)$$

$$\frac{dN}{dt} = N(a_{31}(x) - a_{32}N - a_{33}x - a_{34}I)$$

Where $I(t)$ is the density of infectious disease, $x(t)$ is the density of susceptible prey population and $N(t)$ is the density of predator population in the infected environment at any time $t \geq 0$.

a_{11} : the growth rate coefficient of infectious disease,

a_{21} : the growth rate coefficient of susceptible prey,

$a_{31}(x)$: the specific growth i.e, net birth minus death rate coefficient of predator species. It increases as the density of prey species increases. The function $a_{31}(x)$ may satisfy the following three conditions

1. The predator species may depend partially on the prey species x, i.e x is an alternate food for the predator species. In this case the function $a_{31}(x)$ satisfies the condition $a_{31}(0) = a_0 > 0$, $a_{31}/(0) > 0$ for $x \geq 0$ and large x, let $a_{31}/(x) = K < a_m$.

2. The predator species may wholly depend upon the prey species x. In this case

   $a_{31}(0) = 0$, $a_{31}/(0) > 0$ for $x \geq 0$.

3. The predator species may predating the prey species i.e,

$$a_{31}(0) < 0,\ a_{31}/(0) > 0 \text{ for } x \geq 0.$$

a_{i2} : the intraspecific interference coefficient of the infectious disease, prey and predator population respectively for $i = 1, 2, 3$,

a_{13} : the growth rate coefficient of infectious disease due to the presence of prey species,

a_{14} : the growth rate coefficient of infectious disease due to the presence of predator species,

a_{23} : the feeding rate per predator per unit prey consumed,

a_{24} : the rate of infection to the prey species,

a_{33} : the death rate coefficient of predator species per infected prey consumed,

a_{34} : the rate of infection to the predator species.

All the coefficients are taken to be positive.

Mathematical analysis

Putting $\frac{dz}{dt} = 0$ (for $z = I, x, N$) and solving (1) we get the following non-negative equilibria, namely, $E_0(0,0,0)$, $E_1(a_{11}/a_{12}, 0, 0)$, $E_2(0, a_{21}/a_{22}, 0)$, $E_3(0, 0, a_{31}(x)/a_{32})$, $E_4(\bar{I}, \bar{x}, 0)$, $E_5(0, \bar{\bar{x}}, \bar{\bar{N}})$, $E_6(\bar{\bar{I}}, 0, \bar{N})$, $E^*(I^*, x^*, N^*)$. The equilibria E_0 - E_3 obviously exist, we shall show the existence of the other equilibria in the following:

***Existence of $E_4(\bar{I}, \bar{x}, 0)$*:** In the equilibrium point E_4, the coordinate $\bar{x}$ is given by $\bar{x} = (a_{12}a_{21} - a_{11}a_{24})/(a_{12}a_{22} + a_{13}a_{24})$ which is feasible if we have

$$a_{12}a_{21} > a_{11}a_{24} \qquad (2a)$$

***Existence of $E_5(0, \bar{\bar{x}}, \bar{\bar{N}})$*:** Here $\bar{\bar{x}}$ and $\bar{\bar{N}}$ are the positive solutions of the following equations

$$N = (a_{21} - a_{22}\,x)/a_{23} \qquad (3a)$$

$$N = (a_{31}(x) - a_{33}x)/a_{32} \qquad (3b)$$

From (3a) we note that $\lim_{x\to 0} N = a_{21}/a_{23} > 0,$ (3c)

$$\frac{dN}{dx} = -\, a_{22}/a_{23} < 0 \tag{3d}$$

This shows that N is an decreasing function of x starting from a_{21}/a_{23}.

From (3b) we note that $\lim_{x\to 0} N = a_0/a_{32} > 0$ (3e)

$$\frac{dN}{dx} = (a_{31}'(x) - a_{33})/a_{32} \tag{3f}$$

The two isoclines (3a) and (3b) intersect at a unique point E_5 under the conditions

$$a_{31}'(x) - a_{33} > 0 \tag{4a}$$

and

$$(a_{21}/a_{23}) > (a_{31}(x)/a_{32}) \tag{4b}$$

Existence of $E_6(\bar{I}, 0, \bar{N})$: Here in the equilibrium point E_6, the coordinate $\bar{N}$ is given by

$$N = (a_0 a_{12} - a_{11}a_{34})/(a_{14}a_{34} + a_{12}a_{32}) \tag{5a}$$

Which shows that for equilibrium E_6 to be feasible, we must have

$$a_0 a_{12} - a_{11}a_{34} > 0 \tag{5b}$$

***Existence of $E_7(I^*, x^*, N^*)$* :** I^*, x^* and N^* are the solutions of the following algebraic equations

$$a_{11} - a_{12}I + a_{13}x + a_{14}N = 0 \tag{6a}$$

$$a_{21} - a_{24}I - a_{22}x - a_{32}N = 0 \tag{6b}$$

$$a_{31}(x) - a_{34}I - a_{33}x - a_{32}N = 0 \tag{6c}$$

Solving (6a) and (6b) we get

$$I = \Omega_1/\Omega \text{ and } N = \Omega_2/\Omega \tag{6d}$$

Where $\Omega = a_{12}a_{23} + a_{14}a_{24}$

$$\Omega_1 = (a_{11}a_{23} + a_{14}a_{21}) + (a_{13}a_{23} - a_{14}a_{22})x$$

$$\Omega_2 = (a_{12}a_{21} - a_{11}a_{24}) - (a_{12}a_{22} + a_{13}a_{24})x$$

I^* and N^* exists $\Omega_2 > 0$ (6e)

$$a_{13}a_{23} - a_{14}a_{22} > 0 \tag{6f}$$

along with the condition (2a)

Substituting (6d) in (6c) we get

$$F(x) = a_{31}(x) - a_{34}(\Omega_1/\Omega) - a_{32}(\Omega_2/\Omega) \tag{6g}$$

We note that $F(0) = a_0 - a_{34}(\Omega_1/\Omega) - a_{32}(\Omega_2/\Omega)$,

$$F'(x) = a_{31}/(x) - a_{33}$$

And for x large, $F(x) < 0$,

From (6) we note that $F(0)$, $F'(x)$ may be positive or negative. However, there exists a positive unique solution $x = x^*$ of (6g) in the interval $0 < x^* < \infty$ such that $F(x^*) = 0$, if the following inequalities hold:

$$F(0) > 0, \quad F'(x) < 0, \tag{6h}$$

Knowing the value of x^*, I^* and N^* can be computed from (6d). Thus the interior equilibrium E^* exists if (2a), (6f) and (6h) hold.

Stability of equilibria

The local stability of the equilibria can be studied by computing variational matrices by using Gershgorion's theorem and Routh-Hurwitz criteria (Freedman, 1987) corresponding to each equilibrium. From these matrices we note the following results:

1. As $\lambda_1 = a_{11} > 0$, $\lambda_2 = a_{21} > 0$ and $\lambda_3 = a_0 > 0$, therefore E_0 has unstable manifold in IxN-space.

2. As $\lambda_1 = -a_{12} < 0$, $\lambda_2 = (a_{12}a_{21} - a_{11}a_{24})/a_{12} > 0$ and $\lambda_3 = (a_0 a_{12} - a_{11}a_{34})/a_{12} > 0$ because of conditions (2a) and (5b), thus E1 has stable manifold along I-direction and unstable manifold in the xN-plane.

3. As $\lambda_1 = (a_{11}\, a_{22} + a_{21}\, a_{33})/a_{22} > 0$, E_2 has unstable manifold along I-direction and stable manifold in the xN-plane provided

$$(a_{21}a_{22} + a_{21}a_{33} - a_{22}\, a_{31}(\alpha))/a_{22} > 0$$

and $$a_{21}(a_{21}\, a_{33} + a_{22}\, a'_{31}(\alpha) - a_{22}\, a_{31}(\alpha))/a_{22} > 0 \tag{7a}$$

$$\alpha = a_{21}/a_{22}$$

4. As $\lambda_1 = a_{11} + a_{14}\beta > 0$, $\lambda_2 = a_{21} - a_{23}\beta$, $\lambda_3 = -(a_0 + a_{32}\beta) < 0$, $\beta = a_0/a_{12}$, E_3 has unstable manifold along I-direction and stable in the xN-plane provided $a_{21} - a_{23}\beta < 0$. (7b)

5. E_4 is locally stable if $A_1 A_2 > A_3$ where

$$A_1 = a_{12}\bar{I} + a_{22}\bar{x} - \overline{H}$$

$$A_2 = a_{12}a_{22}\bar{I}\,\bar{x} + a_{22}\bar{x}\,\overline{H} + a_{12}\bar{I}\,\overline{H} + a_{13}a_{24}\bar{I}\,\bar{x} - a_{23}\bar{x}a_{31}(\bar{x})$$

$$A_3 = a_1 a_{22}\bar{I}\,\bar{x}\,\overline{H} - a_{13}a_{24}a_{31}(\bar{x})\bar{I}\,\bar{x} + a_{12}a_{23}a_{31}(\bar{x})\bar{I}\,\bar{x} + a_{13}a_{24}\bar{I}\,\bar{x}$$

$$\overline{H} = a_{31}(\bar{x}) - a_{34}\bar{I} - a_{33}\bar{x}$$

6. As $\lambda = a_{11} + a_{33}\bar{\bar{x}} + a_{14}\bar{\bar{\bar{N}}} > 0$, $B_1 = a_{22}\bar{\bar{x}} + a_{32}\bar{\bar{\bar{N}}} > 0$ and

$B_2 = a_{33}\bar{\bar{x}}\, a_{14}\bar{\bar{\bar{N}}} + a_{23}\bar{\bar{x}}\,(a'_{33}(\bar{\bar{x}}) - a_{14}\bar{\bar{\bar{N}}})$ E_5 has unstable manifold along I-direction and stable manifold in the xN-plane provided $B_2 > 0$

7. As $\lambda = a_{21} - a_{23}\overleftrightarrow{N} - a_{24}\overleftrightarrow{I}$, $G_1 = a_{32}\overleftrightarrow{N} + a_{12}\overleftrightarrow{I} > 0$ and $G_2 = a_{12}a_{32}\overleftrightarrow{I}\,\overleftrightarrow{N} + a_{32}a_{14}\overleftrightarrow{I}\,\overleftrightarrow{N} > 0$, E_6 has stable manifold in *IxN*-space provided $\lambda > 0$.

8. If the inequalities L_i $(i = 2,3) > 0$ and $L_1L_2 > L_3$ hold where

$$L_1 = a_{12}I^* + a_{22}x^* + a_{32}N^* > 0,$$

$$L_2 = (a_{12}a_{22} + a_{12}a_{32} + a_{13}a_{24})I^*x^* + a_{22}a_{32}x^*N^* + a_{23}\{a'_{31}(x^*) - a_{33}N^*\} + a_{14}a_{34}I^*N^*.$$

$$L_3 = \{a_{12}\,a_{22}\,a_{32} - a_{13}\,a_{24}\,a_{32} - a_{14}\,a_{22}\,a_{34} + a_{13}\,a_{23}\,a_{34}\}I^*x^*N^* + (a_{14}\,a_{24} + a_{12}a_{23})\{a'_{31}(x^*) - a_{33}N^*\}x^*N^*.$$

Then E_7^* is locally asymptotically stable.

In general there is no obvious remarks to be made about the stability of E_7^* In the following theorem we state sufficient condition for E_7^* to be locally asymptotically stable.

Theorem 1: Let the following inequality holds,

$$[(a'_{31}(x^*) + a_{33})\,a_{14}a_{24} + a_{23}a_{13}a_{34}]^2 < a_{13}a_{14}a_{22}a_{24}a_{32}a_{34} \qquad (8)$$

then E_7^* is locally asymptotically stable.

Proof: Linearizing the system (1) by substituting

$I = I^* + i$, $x = x^* + X$, $N = N^* + n$ and using the following Liapunov's function

$$U(I,X,n) = [i^2/I^* + a_{13}\,X^2/(a_{24}x^*) + a_{14}\,n^2/(a_{34}N^*)]/2 \qquad (8a)$$

It can be checked that the time derivative of U along the solutions of system (1) under condition (8) is negative definite, proving the theorem.

In order to investigate the global stability behavior of the interior equilibrium E_7^*, we first state the following Lemma which establishes a region of attraction for the system (1). The proof of the lemma is easy and hence is omitted.

Lemma 1: The set $\Omega = \{(I, x, N) : 0 \leq I \leq M_i, 0 \leq x \leq M_x, 0 \leq N \leq M_n\}$ attracts all solutions initiating in the positive orthant, where

$$M_i = [a_{11}a_{22}a_{32} + a_{13}a_{21}a_{32} + a_{14}a_{22}a_{31}(\alpha)]/a_{12}a_{22}a_{32}$$

$$M_x = a_{21}/a_{22}$$

$$M_n = a_{31}(\alpha)/a_{32}$$

Theorem 2: In addition to the assumption $a_{31}(0) = a_0 > 0$, let $a_{31}(x)$ satisfies the condition

$$0 \leq a'_{31}(x) \leq a_m \tag{9}$$

in Ω for some positive constant a_m, if the following inequality holds

$$[(a_m + a_{33})a_{14}a_{24} + a_{13}a_{23}a_{34}]^2 < a_{13}a_{14}a_{22}a_{24}a_{32}a_{34} \tag{9a}$$

Then E_7^* is globally asymptotically stable with respect to all solutions initially in the positive orthant.

Proof: Let us consider the following positive definite function around E_7^*

$$V(I,x,N) = I - I^* - I^*\ln(I/I^*) + c_1\{x - x^* - x^* \ln(x/x^*)\} + c_2(N - N^* - N^*\ln(N/N^*)) \tag{9b}$$

where $c_1 = a_{13}/a_{24}$, $c_2 = a_{14}/a_{34}$.

Differentiating V with respect to t along the solutions of (1), a little manipulation yields

$$\dot{V}(I,x,N) = -a_{12}(I - I^*)^2 - c_1a_{22}(x - x^*)^2 - c_2a_{32}(N - N^*)^2 + [(\gamma - a_{33})\, a_{14}a_{24} - a_{23}a_{13}a_{34}]/a_{24}a_{34}$$

Where $\gamma(x) = \{a_{31}(x) - a_{31}(x^*)\}/(x - x^*)$ when $x = x^*$

and $a'(x^*)$ when $x \neq x^*$

we note from (9) and the mean value theorem that $|\gamma(x)| \leq a_m$.

Sufficient condition for V to be negative definite that

$$[(a_m + a_{33})\, a_{14}a_{24} + a_{23}a_{13}a_{34}]^2 < a_{13}a_{14}a_{22}a_{24}a_{32}a_{34}$$

Proving the theorem.

(1) From the above discussion we note that local stability implies the global stability to the system.

(2) Global stability depends on the intraspecific interference of prey and predator species but not on the infectious disease.

(3) It is also noted that the global stability depends on the feeding rate per predator per unit of infected prey consumed. If the feding rate is too high then the system may be unstable.

(4) It is also found that if the rate of infection is not controlled in proper time then spread of an infection disease may lead to a dramatic drop in prey as well as it affects the predator population with the entire ecosystem.

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