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STOKES FLOW PAST A SPHERE WITH A TIME DEPENDENT SOURCE AT ITS CENTRE

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ABSTRACT

The aim of the present paper is to investigate the influence of a time dependent source on the Stokes flow past a sphere. The effect manifests itself through the source parameter $s(=Q(t)/\nu a)$, where $Q(t)$ represents the source strength and the analysis is valid for $Re < s < 1$, Re being the Reynolds number. A special case $s = s_0(1-e^{-\alpha t})$ is studied in detail and it is found that while incipiently the drag gets increased, ultimately it is reduced by the source.

1. *Introduction* :

The theory of slow flow was initiated by Stokes paper [4] where the motion produced by small rectilinear oscillation of a sphere in a fluid at rest is investigated. The problem is solved by neglecting the inertia term under the assumption that it is small because of small velocities involved. In the present study the unsteady motion of a fluid of kinematic viscosity ν about a pervious sphere of radius a is investigated. The initial motion is the Stokes flow due to a uniform stream U and the unsteadiness is generated by a time dependent source Q placed at its centre. Far away from the sphere the source flow is neglegible so that the velocity is still that of a uniform stream. It can be seen that if Q is large enough so that the non-dimensional source parameter s $(=Q/\nu a)$ is not negligible, the inertia term being non-linear cannot be altogether omitted; the equation, however, can still be linearized if the Reynolds number Re $(=Ua/\nu)$ is small as compared to s. This assumption which amounts to neglecting terms of order U^2 while retaining terms of order QU is justifiable at least in the vicinity of the sphere where the Stokes approximation is valid too.

2. *Basic Equations and Simplification* :

The equations of momentum and continuity governing the fluid flow are

$$\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}.\ \text{grad}\ \mathbf{V}=-\frac{1}{\rho}\ \text{grad}\ p+\nu\nabla^2\mathbf{V},$$

$$\text{div}\ \mathbf{V}=0,$$

where ρ is the density, p the pressure and $\mathbf{V}(v_r,\ v_\theta,\ v_\phi)$ the velocity; $r,\ \theta,\ \phi$ being spherical polar coordinates with the pole at the centre of the sphere and axis along the direction of the uniform stream.

On account of the axial symmetry v_ϕ is taken to be zero and then the following transformation

$$r=ar',\ \ t=\frac{a^2t'}{\nu},\ \ p=\frac{\rho\nu U}{a}\{p_0(r',\ t')+p_1(r',\ t')\cos\theta\},$$

$$v_r=\frac{Q(t')}{a^2r'2}+U\{u(r',\ t')+1\}\cos\theta,$$

$$v_\theta=U\{v(r',\ t')-1\}\sin\theta,$$

is made to non-dimensionalize the equations which can then be linearized by assuming as in Stokes approximation that U is small and the quantities involving square of it are negligible; dropping the accent we then get

$$(1)\qquad \left[\begin{aligned} &\frac{\partial u}{dt}+\frac{s(t)}{r^2}\left(\frac{\partial u}{\partial r}-\frac{2(u+1)}{r}\right)=-\frac{\partial p_1}{\partial r}+\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r},\\ &\frac{\partial v}{\partial t}+\frac{s(t)}{r^2}\left(\frac{\partial v}{\partial r}+\frac{v-1}{r}\right)=\frac{p_1}{r}+\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}-\frac{2(u+v)}{r^2},\\ &\frac{\partial u}{\partial r}+\frac{2}{r}(u+v)=0,\end{aligned}\right.$$

where $s(t)=Q(t)/\upsilon a$ is the parameter determining the effect of the source. The equation for p_0 has not been included above as it does not affect the drag.

The above equations are to be solved subject to the following initial and boundary conditions

$$(2)\quad \left[\begin{array}{l} \left.\begin{array}{l} u(r,0)=u_0=-\dfrac{3}{2r}+\dfrac{1}{2r^3}, \\ v(r,\theta)=v_0=\dfrac{3}{4r}+\dfrac{1}{4r^3} \end{array}\right\} r>1, \\ \text{and} \\ \left.\begin{array}{l} u(1,t)=-v(1,t)=-1, \\ u,\ v\to 0 \text{ as } r\to\infty \end{array}\right\} t>0, \end{array}\right.$$

where u_0, v_0, represent the initial Stokes flow.

Eliminating p_1 and v inbetween equations (1) and then suitably integrating twice, we get

$$(3)\quad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\frac{\partial u}{\partial r}-s(t)\left(\frac{1}{r^2}\frac{\partial u}{\partial r}+\frac{4u}{r^3}+\int\frac{6u}{r^4}dr\right)+\frac{c(t)}{r^3},$$

where the other constant of integration has been put equal to zero as far away from the sphere the contribution of the source dies down and so $u\to u_0$, and with this value the terms occuring in (3) all tend to zero as $r\to\infty$.

Now, by making the transformation

$$(4)\quad \eta=\frac{1}{r}\frac{\partial}{\partial r}(r^3u)+3,$$

the equation (3) can be put in the form

$$(5)\quad \frac{\partial\eta}{\partial t}=\frac{\partial^2\eta}{\partial r^2}-\frac{s(t)}{r^2}\left\{\frac{\partial\eta}{\partial r}+3r^3\int\frac{\eta dr}{r^5}+\frac{9}{4r}\right\},$$

which is to be solved under the following conditions

$$(6)\quad \left[\begin{array}{l} \eta(r,0)=0,\quad r>1, \\ \left.\begin{array}{l}\eta(1,t)=0 \\ \qquad \eta\to 0 \text{ as } r\to\infty\end{array}\right\} t>0. \end{array}\right.$$

Since in the absence of the source $(s=0)$, $u=u_0$ and $\eta=0$, therefore, for small s, u and η can be approximated by u_0 and zero respectively in the above equations. Thus, under this approximation, equation (5) reduces to

$$(7)\quad \frac{\partial^2\eta}{\partial r^2}-\frac{\partial\eta}{\partial t}=\frac{9}{4r^3}s(t).$$

Also, by the help of equations (1), (3), (4) and (7) we get

$$(8)\qquad \left[\begin{aligned} u &= \frac{1}{r_3}\int_1^r x\eta(x, t)dx - \frac{1}{2}(3r^2-1), \\ p_1 &= \frac{c(t)}{2r^2} - s(t)\left\{\frac{1}{r^2} - \frac{9}{4r^3} + \frac{1}{2r^5}\right\}, \\ c(t) &= -\left(\frac{\partial\eta}{\partial r}\right)_{r=1} - 3 - \frac{9}{4}s(t). \end{aligned}\right.$$

2. *Solution :*

The solution of equation (7) can be obtained by an application of the method of Laplace transform [2]. The transformed equation is

$$(9)\qquad \frac{d^2\bar{\eta}}{dr^2} - p\bar{\eta} = \frac{9}{4r_3}\bar{s}(p),$$

where

$$\bar{\eta}(r, p) = \int_0^\infty e^{-pt}\eta(r, t)\, dt \text{ and } \bar{s}(p) = \int_0^\infty e^{-pt}s(t)\, dt.$$

The solution of equation (9) subject to the transformed boundary conditions

$$(10)\qquad \left[\begin{aligned} &\bar{\eta}(r, p) = 0, \text{ at } r = 1, \\ &\bar{\eta}(r, p) \to 0 \text{ as } r \to \infty, \end{aligned}\right.$$

is

$$(11)\qquad \bar{\eta}(r, p) = \frac{9}{16}\bar{s}(p)\Big[\frac{2}{r} - \sqrt{p}\left\{e^{r\sqrt{p}}E_1(r\sqrt{p}) + e^{-r\sqrt{p}}E_i(r\sqrt{p})\right\} - e^{-(r-1)\sqrt{p}}\left\{2 - \sqrt{p}\left(e^{\sqrt{p}}E_1(\sqrt{p}) + e^{-\sqrt{p}}E_i(\sqrt{p})\right)\right\}\Big],$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt \text{ and } E_i(z) = \int_{-\infty}^z \frac{e^t}{t}dt$$

are singled valued function in the z-plane cut along the negative real axis.

The function $\eta(r, t)$ can be obtained from $\bar{\eta}(r, p)$ by the well known inversion formula

$$(12)\qquad \eta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\bar{\eta}(r, p)\, dp.$$

Explicit solution in the special case $s(t)=s_0(1-e^{-\alpha t})$, $(\alpha>0)$ will now be given. For finite α this describes a source which strating from zero initial value establishes itself to ultimate value s_0; and for $\alpha \to \infty$ corresponds to an impulsive source since initially the flow corresponds to Stokes flow. The corresponding value $\bar{s}(p)=\frac{s_0\alpha}{p(p+\alpha)}$ when substituted in (11) shows that the integrand in (12) has a branch point at $p=0$ and a single pole at $p=-\alpha$; therefore, the integral can be evaluated by the help of the closed contour C along the negative real axis and indented above and below $-\alpha$, as shown in Fig. 1, we thus obtain

$$(13)\quad \eta(r,t)=\frac{9}{8}s_0\Big[\frac{1}{r}(1-e^{-\alpha t})-\alpha e^{-\alpha t}\int_0^t e^{\alpha\lambda}\operatorname{erfc}\frac{r-1}{2\sqrt{\lambda}}d\lambda$$

$$+\sqrt{\alpha}e^{-\alpha t}\Big\{F(r\sqrt{\alpha})=\frac{\pi}{2}\sin(r-1)\sqrt{\alpha}\sin\sqrt{\alpha}$$

$$-\cos(r-1)\sqrt{\alpha}\,F(\sqrt{\alpha})\Big\}+\frac{2\alpha}{\pi}\int_0^\infty\frac{e^{-u^2t}}{\alpha-u^2}\Big\{\frac{\pi}{2}\cos u$$

$$+F(u)\Big\}\sin(r-1)u\,du\Big],$$

and so

$$(14)\quad \left(\frac{\partial\eta}{\partial r}\right)_{r=1}=\frac{9}{8}s_0\Big[e^{-\alpha t}-1+2\sqrt{\frac{\alpha}{\pi}}D(\sqrt{\alpha t})+\alpha e^{-\alpha t}\Big\{G(\sqrt{\alpha})-\frac{\pi}{2}\sin\sqrt{\alpha}\Big\}$$

$$+\frac{2\alpha}{\pi}\int_0^\infty\frac{ue^{-u_2t}}{\alpha-u^2}\Big\{\frac{\pi}{2}\cos u+F(u)\Big\}du\Big],$$

where [1]

$$F(z)=Ci(z)\sin z-Si(z)\cos z,$$

$$G(z)=Ci(z)\cos z+Si(z)\sin z,$$

$$Ci(z)=-\int_z^\infty\frac{\cos t}{t}dt,\quad Si(z)=\int_0^z\frac{\sin t}{t}dt$$

and $D(z)$ is the Dawson's integral [1, p, 319] $e^{-z^2}\int_0^z e^{t^2}dt$.

In order to calculate the drag the Cauchy's principal value of the Improper integral occuring in (14) has to be evaluated numerically, but useful estimates [2] for small and large values of time can be made by using the asymptotic expansions and series expansions for large and small values of p respectively.

Thus for large p, using the asymptotic expansions for $E_1(p)$ and E_1 (p) [1], we have

$$\bar{\eta}(r, p) \sim \frac{9}{4}\frac{\bar{s}(p)}{p}\left[e^{-(r-1)\sqrt{p}} - \frac{1}{r^3}\right]$$

and so for small values of time, we get on using convolution theorem [3]

$$\eta(r, t) \sim \frac{9}{4}\int_0^t s(t-\tau)\left\{\operatorname{erfc}\frac{r-1}{2\sqrt{\tau}} - \frac{1}{r^3}\right\}d\tau.$$

This reduces for the case $s(t) = s_0(1-e^{-\alpha t})$, to

$$\eta(r, t) \sim \frac{9}{4}s_0\left[\frac{1}{r^3}\left\{\frac{1}{\alpha}(1-e^{-\alpha t}) - t\right\} + \left(t + \frac{(r-1)^2}{2}\right)\operatorname{erfc}\frac{r-1}{2\sqrt{t}}\right.$$

$$\left. -(r-1)\sqrt{\frac{t}{\pi}}\, e^{-\frac{(r-1)^2}{4t}} - \frac{1}{\alpha}(1-e^{-\alpha t}) + e^{-\alpha t}\int_0^t e^{\alpha\tau}\operatorname{erf}\frac{r-1}{2\sqrt{\tau}}\,d\tau\right]$$

and so we have

$$(15)\qquad \left(\frac{\partial\eta}{\partial r}\right)_{r=1} \sim \frac{9}{4}s_0\left[3\left\{t - \frac{1}{\alpha}(1-e^{-\alpha t})\right\} - 2\sqrt{\frac{t}{\pi}} + \frac{2}{\sqrt{\pi\alpha}}D(\sqrt{\alpha t})\right],$$

which for $\alpha \to \infty$ becomes

$$(16)\qquad \left(\frac{\partial\eta}{\partial r}\right)_{r=1} \sim \frac{9}{4}s_0\left(3t - 2\sqrt{\frac{t}{\pi}}\right).$$

Similarly using series expansions for $E_1(p)$ and $E_i(p)$ [1] we have for large values of time

$$(17)\qquad \left(\frac{\partial\eta}{\partial r}\right)_{r=1} = \frac{9}{8}s_0\left(-1 + \frac{1}{\sqrt{\pi t}} - \frac{1}{2t}\right).$$

4. *Drag :*

Now the drag force experienced by the sphere can be calculated for small s as follows

$$D = 2\pi a^2\int_0^\pi\left[\left(-p + 2\rho\nu\frac{\partial v_r}{\partial r}\right)\cos\theta - \rho\nu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right)\sin\theta\right]_{r=a}\sin\theta\, d\theta$$

$$(18)\qquad = 6\pi\rho_\nu Ua\left[1 + \frac{1}{3}\left(\frac{\partial\eta}{\partial r}\right)_{r=1} + \frac{s(t)}{12}\right].$$

It is easily seen on taking into account equation (17) that for large t, the drag remains unaffected by α. The following table shows that variation of drag coefficient $D_c = D/6\pi\rho\nu Ua$ for small and large time when $s_0 = 1$.

Table 1

	for small time			for large time
	Drag coefficient D_c			Drag coefficient D_c
$\sqrt{t}$	$\sqrt{\alpha}=10$	$\sqrt{\alpha}\to\infty$	$\sqrt{t}$	(independent of α)
0.00	1.0000	1		
0.02	1.0028	1.0673	5	0.7431
0.04	1.0092	1.0531	10	0.7276
0.06	1.0159	1.0380	15	0.7217
0.08	1.0205	1.0325	20	0.7184
0.10	1.0219	1.0212	25	0.7165
0.12	1.0202	1.0142	30	0.7145
0.14	1.0165	1.0090	40	0.7135
0.16	1.0121	1.0056	50	0.7125
0.18	1.0075	1.0039	100	0.7104
0.20	1.0060	1.0041	200	0.7094
			∞	0.7083

It is interesting to note that while for small time D_c gets increased on account of source, it is reduced for large time attaining its ultimate value 0.7083. When t is small, the behaviour for $\sqrt{\alpha}=10$ and $\sqrt{\alpha}\to\infty$ is also widely different. In the first case there is an increase upto a certain maximum value followed by a decrease which should be there as ultimately $D_c<1$. In the second situation the maximum increase occurs immediately at the start of the source and then there is a gradual decrease upto $\sqrt{t}=1/3\sqrt{\pi}$. As the ultimate value of D_c is less than unity this turning point suggests that for this case the small time analysis breaks down here. The different behaviour for α finite and α infinite is expected as the sources described by them are quite different ; at the start while the source strength is zero for the former, it is at its maximum value s_0 for the latter.

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