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UNSTEADY FLOW THROUGH A POROUS MEDIUM BETWEEN TWO PARALLEL FLAT PLATES

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ABSTRACT

Aim of this paper is to study the unsteady flow of a viscous liquid through a porous medium between two parallel flat plates under the influence of pressure gradient, (i) varying linearly with time, and (ii)decreasing exponentially with time using the generalized momentum equation. In the first case it is seen that the symmetrical points have the same velocity.

1. *Introduction* :

Flow of a viscous liquid in a porous medium is of great importance in the study of percolation through soils in hydrology, petroleum industry and in agricultural engineering, etc. Flows of different fluids through various types of porous media are studied employing the classical Darcy's law which states that the seepage velocity of the fluid is proportional to the pressure gradient. This law fails to explain the phenomena occuring in highly porous media such as fibreglass. The viscous stress at the surface is able to penetrate into the medium and produces the flow near the surface even in the absence of the pressure gradient. Brinkman (1947) generalized the Darcy law taking into account the effect of viscous stress. Brinkman's law gave good results in the case of highly porous media. Yamamoto (1971, 1973) investigated the flow past porous bodies using the generalized law.

The present paper consists of two parts. In part A the flow through a porous medium between two parallel flat plates under pressure gradient varying linearly with time is discussed. An expression for the velocity is obtained in dimensionless form. This consists of two parts, the one varies linearly with the parameter $T=\frac{\nu t}{y_0{}^2}$ and the other is the transient part of the velocity which vanishes in the limit as t

tends to infinity. It is seen that the contribution of the transient part is insignificant when $T > 1.5$.

In part B the flow of a viscous liquid through a porous medium between two parallel flat plates under exponentially decreasing pressure gradient is studied. An expression for the velocity has been obtained taking

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=a_0+\sum_{m=1}^{\infty} a_m e^{-n_m t}.$$

which has been compared with that of Dube's result (1969) where he has obtained the velocity of a viscous liquid in a channel bounded by two parallel flat plates under exponentially decreasing pressure gradient.

2. *Equations of Motion* :

The equations of motion of a viscous liquid through a porous medium as proposed by Brinkman is

$$\frac{\partial \mathbf{q}}{\partial t}+\left(\mathbf{q}\cdot\nabla\right)\mathbf{q}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{q}-\frac{\nu}{k}\mathbf{q}, \tag{1}$$

where $\mathbf{q}$ is the velocity vector whose components are u, v and w parallel to the axes respectively ρ the density of the liquid, ν, k are the kinematic co-efficient of viscosity of the liquid and the permeability constant of the medium respectively. The equation of continuity is

$$div\ \mathbf{q}=0. \tag{2}$$

for the present problem we have

$$\left.\begin{array}{l} u=u(x,y,t),\ v=0,\ w=0, \\ P=P(x,y,t),\ \dfrac{\partial}{\partial 3}(\quad)=0, \end{array}\right\} \tag{3}$$

The last equation holds because the motion is two-dimensional. Furthermore, the equation of continuity (2) and the conditions (3) give

$$\frac{\partial u}{\partial x}=0 \text{ so that } u=u(y,t) \tag{4}$$

Substituting equations (3) and (4) into the equations of motion (1), we have

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{k}. \tag{5}$$

and

$$\frac{\partial P}{\partial y}=0 \text{ or } P=P(x, t) \tag{6}$$

From equations (5) and (6) we see that $\frac{\partial P}{\partial x}$ must be a constant or a function of time only in the present problem because P is not a function of y and u is not a function of x.

PART A

3. *Pressure gradient varies linearly with time* :

We now assume that

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=a_0+at. \tag{7}$$

Equation (5) then becomes

$$\frac{\partial u}{\partial t}=a_3+at+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{k}u. \tag{8}$$

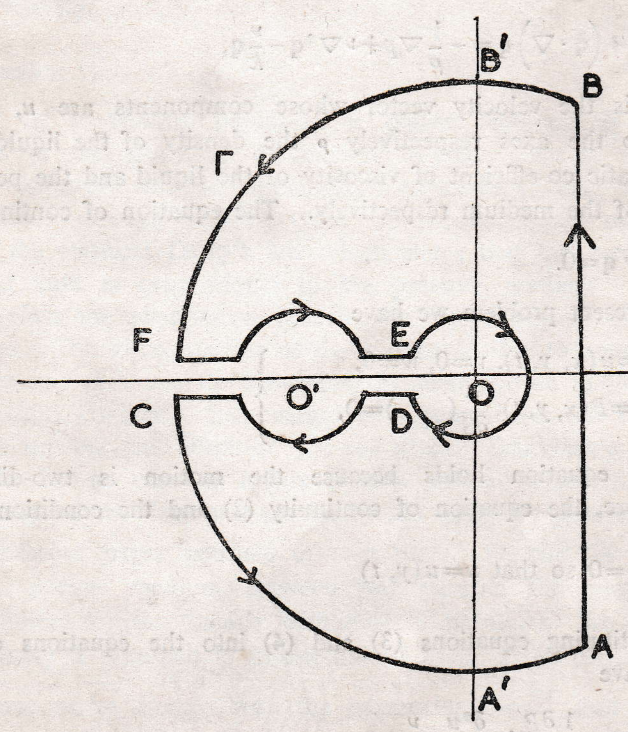

Fig. 1

Let $\overline{u}=\int_0^\infty e^{-st} u\,dt$ be the Laplace transform of u and let u_0 be the initial value of u.

Multiplying equation (8) by exp. $(-st)$ and integrating between the limits 0 to ∞, we get

$$\frac{d^2\overline{u}}{dy^2}-p^2\overline{u}=-\frac{1}{\upsilon}\left[u_0+\frac{a_0}{s}+\frac{a}{s^2}\right], \tag{9}$$

where $p^2=\frac{1}{\nu}\left\{s+\frac{\nu}{k}\right\}$.

We shall now find u_0.

Initially the pressure gradient is a_0 and the motion is steady in the channel.

Hence $\frac{\partial u_0}{\partial t}=0$ and we obtain

$$\frac{d^2u_0}{dy^2}-\frac{1}{k}u_0=-\frac{a_0}{\nu}. \tag{10}$$

The boundary conditions are

$u_0=0$ when $y=-y_0$,

and

$u_0=0$ when $y=y_0$.

The solution of equation (10) under the above boundary conditions are

$$u_0=\frac{ka_0}{\nu}\left[1-\frac{\cosh\left\{\frac{y}{\sqrt{k}}\right\}}{\cosh\left\{\frac{y_0}{\sqrt{k}}\right\}}\right].$$

Substituting this value of u_0 in (9), we get

$$\frac{d^2\overline{u}}{dy^2}-p^2\overline{u}=-\frac{1}{\upsilon}\left[\frac{ka_0}{\nu}\left\{1-\frac{\cosh\left\{\frac{y}{\sqrt{k}}\right\}}{\cosh\left\{\frac{y_0}{\sqrt{k}}\right\}}\right\}\frac{a_0}{s}+\frac{a}{s^2}\right]. \tag{11}$$

The boundary conditions for $\overline{u}$ are

$\overline{u}=0$ when $y=-y_0$,

and

$\overline{u}=0$ when $y=y_0$.

The solution of equation (11) under the above boundary conditions is

$$\bar{u}=\frac{ka_0}{\nu s}\left[1-\frac{\cosh\left\{\frac{y}{\sqrt{k}}\right\}}{\cosh\left\{\frac{y_0}{\sqrt{k}}\right\}}\right]$$

$$+\frac{a}{s^2\left\{s+\frac{\nu}{k}\right\}}\left[1-\frac{\cosh\, py.}{\cosh\, py_0}\right].$$

Now applying Laplace inversion theorem, we get

$$u=\frac{ka_0}{\nu}\left[1-\frac{\cosh\left\{\frac{y}{\sqrt{k}}\right\}}{\cosh\left\{\frac{y_0}{\sqrt{k}}\right\}}\right]$$

$$+\frac{ka}{\upsilon}\left\{t-\frac{k}{\upsilon}\right\}\left[1-\frac{\cosh\left\{\frac{y}{\sqrt{k}}\right\}}{\cosh\left\{\frac{y_0}{\sqrt{k}}\right\}}\right]$$

$$-\frac{ak^{\frac{3}{2}}}{2\upsilon^2\cosh\left(\frac{y_0}{\sqrt{k}}\right)}\left[\left\{y-y_0\tan h\left(\frac{y_0}{\sqrt{k}}\right)\cot h\left(\frac{y}{\sqrt{k}}\right)\right\}\sin h\left(\frac{y}{\sqrt{k}}\right)\right]$$

$$+\frac{4a}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n exp\left[-\left\{\frac{(2n+1^2)\pi^2\nu t}{4y_0{}^2}+\frac{\upsilon t}{k}\right\}\right].\cos\left[\frac{(2n+1)\pi y}{2y_0}\right]}{(2n+1\left[\frac{(2n+1)^2\pi^2\nu}{4y^2{}_0}+\frac{\nu}{k}\right]^2}. \quad (12)$$

Now we make equation (12) dimensionless by introducing

$$U=\frac{u}{U_o},\ \frac{y}{y_0}=\gamma,\ T=\frac{\nu t}{y_0{}^2},$$

where U_o is a characteristic velocity.

We then get

$$U=2b_o k_1\left[1-\frac{\cosh\left(\frac{\gamma}{\sqrt{k_1}}\right)}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\right]$$

$$+2bk_1(T-k_1)\left[1-\frac{\cosh\left(\frac{\gamma}{\sqrt{k_1}}\right)}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\right]$$

$$-\frac{bk_1^{\frac{3}{2}}}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\left[\left\{\gamma-\tanh\left(\frac{1}{\sqrt{k_1}}\right)\cot h\left(\frac{\gamma}{\sqrt{k_1}}\right)\right\}\sin h\left(\frac{\gamma}{\sqrt{k_1}}\right)\right]$$

$$+\frac{128b}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\ exp}{(2n+1)}\frac{\left[-\left\{\frac{(2n+1)^2\pi^2T}{4}+\frac{T}{k_1}\right\}\right].\cos\left(\frac{(2n+1)\pi\gamma}{2}\right)}{\left[(2n+1)^2\pi^2+\frac{4}{k_1}\right]^2}, \quad (13)$$

where

$$b_0=\frac{a_0y_0^3}{2\nu U_0}, \text{ a non-dimensional number,}$$

$$b=\frac{ay_0^4}{2\nu^2U_0}, \text{ a non-dimensional number,}$$

$$K_1=\frac{K}{y_0^2}, \text{ the permeability number.}$$

We now take $U=U_1+U_2$, where

$$U_1=2b_0K_1\left[1-\frac{\cosh\left(\frac{\gamma}{\sqrt{k_1}}\right)}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\right]$$

$$+2bk_1(T-K_1)\left[1-\frac{\cosh\left(\frac{\gamma}{\sqrt{k_1}}\right)}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\right]$$

$$-\frac{b\,k_1^{\frac{3}{2}}}{\cosh\left(\frac{1}{\sqrt{k_1}}\right)}\left[\left\{\gamma-\tanh\left(\frac{1}{\sqrt{k_1}}\right)\ \coth\left(\frac{\gamma}{\sqrt{k_1}}\right)\right\}\ \sinh\left(\frac{1}{\sqrt{k_1}}\right)\right]$$

and

$$U_2=\frac{128\,b}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n e\times p\left[-\left\{\frac{(2n+1)^2\pi^2T}{4}+\frac{T}{K_1}\right\}\right].\cos\left[\frac{(2n+1)\pi r}{2}\right].}{(2n+1)\left[(2n+1)^2\pi^2+\frac{4}{K_1}\right]^2}$$

The values of U for different values of γ and T have been tabulated below when $b_0=4,\ b=\frac{1}{4}$.

Table I. $K_1=9$

γ \ T	0.5	1.0	1.5
0.0	1.7846	1.8828	1.9948
0.4	3.2302	3.3140	3.4090
0.8	1.4439	1.4830	1.5264

Table II. $K_1=16$

γ \ T	0.5	1.0	1.5
0.0	0.1499	0.2518	0.3682
0.4	3.3189	3.4051	3.5029
0.8	1.4291	1.4673	1.5099

Table III. $K_1=25$

γ \ T	0.5	1.0	1.5
0.0	-2.0643	-1.9618	-1.8445
0.4	3.3471	3.4335	3.5319
0.8	1.4371	1.4751	1.5177

The values of U for negative values of γ have not been given here because of the fact that the velocity will not change whether γ is negative or positive. It means that the symmetrical points have the same velocity. The value of U beyond $T=1.5$ have also not been given because U_2 is very small compared to U_1 when $T>1.5$, hence the transient part is insignificant and U varies linearly with T in this range. From the tables it is observed that U increases with T for fixed γ. It is also seen from these tables that the maximum of U does not occur at the axis of the channel, but is shifted towards the walls in the present case.

PART B

4. *Pressure gradient decreases exponentially with time :*

We take

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}=a_0+\sum_{m=1}^{\infty}a_m e^{-mt}. \tag{14}$$

Equation (5) then becomes

$$\frac{\partial u}{\partial t}=a_0+\sum_{m=1}^{\infty}a_m e^{-mt}+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{k}u. \tag{15}$$

Let $\bar{u}=\int_0^{\infty}e^{-st}udt$ be the Laplace transform of u and let u_0 be the initial value of u. Multiplying equation (15) by $exp(-st)$ and integrating between the limits 0 to ∞, we get

$$\frac{d^2\bar{u}}{dy^2}-p^2\bar{u}=-\frac{1}{\upsilon}\left[u_0+\frac{a_0}{s}+\sum_{m=1}^{\infty}\frac{a_m}{(s+m)}\right], \tag{16}$$

where

$$p^2=\frac{1}{\upsilon}\left(s+\frac{\upsilon}{k}\right).$$

Here again

$$u_0=\frac{Ka_0}{\nu}\left[1-\frac{\cosh\left(\frac{y}{\sqrt{K}}\right)}{\cosh\left(\frac{y_0}{\sqrt{K}}\right)}\right].$$

The solution of the equation (16) under the boundary conditions

$$\bar{u}=0 \quad \text{when} \quad y=-y_0,$$

and

$$\bar{u}=0 \quad \text{when} \quad y=y_0$$

is

$$\bar{u}=\frac{Ka_0}{s\nu}\left[1-\frac{\cosh\left(\frac{y}{\sqrt{K}}\right)}{\cosh\left(\frac{y_0}{\sqrt{K}}\right)}\right]$$

$$+\frac{1}{\left(s+\frac{\nu}{k}\right)}\left[1-\frac{\cosh py}{\cosh py_0}\right]\sum_{m=1}^{\infty}\frac{a_m}{(s+m)}. \tag{17}$$

Now applying Laplace inversion theorem we get

$$u=\frac{Ka_0}{\nu}\left[1-\frac{\cosh\left(\frac{y}{\sqrt{K}}\right)}{\cosh\left(\frac{y_0}{\sqrt{K}}\right)}\right]$$

$$-\sum_{m=1}^{\infty}\frac{a_m}{\left(m-\frac{\nu}{K}\right)}\left[1-\frac{\cos\left\{\frac{1}{\sqrt{\nu}}\left(m-\frac{\nu}{K}\right)^{\frac{1}{2}}y\right\}}{\cos\left\{\frac{1}{\sqrt{\nu}}\left(m-\frac{\nu}{K}\right)^{\frac{1}{2}}y_0\right\}}\right]e^{-mt}$$

$$+\frac{4}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a_m \, exp\left[-\left\{\frac{(2n+1)^2\pi^2\nu t}{4y_0^{\,2}}+\frac{\nu t}{K}\right\}\right].\cos\left[\frac{(2n+1)}{2y_0}\pi y\right]}{(2n+1)\left[m-\left\{\frac{(2n+1)^2\pi^2\nu}{4y_0^{\,2}}+\frac{\nu}{K}\right\}\right]}. \tag{18}$$

The expression (18) for the velocity is similar to the expression obtained by Dube (1969) where he has found the velocity of a viscous liquid in a channel bounded by two parallel flat plates under exponentially decreasing pressure gradient.

Also, (13) and (18) reduce to the usual laminar flow between two parallel flat plates discussed by Dube for $K_1 \to \infty \, (K \to \infty)$.

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