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SIMULTANEOUS DIFFUSION AND MASS FLOW TO PLANT ROOTS-II—A THEORETICAL SOLUTION

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ABSTRACT

We present here a theoretical solution of the mathematical model for simultaneous diffusion and mass flow to plant roots, given by P. H. Nye and J. A. Pspiers, [1964] in the following form :—

$$\frac{1}{r}\frac{\partial}{\partial r}\left[D'r\frac{\partial C}{\partial r}+\gamma_0 r_0 C\right]=\frac{\partial C'}{\partial t}$$

under the assumption that $C'=C$. P. H. Nye, (1964) gave the solution under the assumption that $\frac{\partial C'}{\partial t}=0$ [constants involved are defined on page (36)].

1. *Introduction* :

The greater part of the substances entering a plant root have to move to the root surface either by diffusion, or by mass flow in the solution sucked towards the root by transpiration. Both processes usually occur together, and their relative importance varies widely with such factors as the diffusion co-efficient of the substance, the rate of transpiration and the geometry of the root. Substance of interest include not only nutrients, but also pesticides, toxins and stimulants.

Bouldin, [1961] had discussed theoretically the diffusion process on its own. Passioura, [1963] has treated simultaneous diffusion and mass flow by regarding the two processes as occuring separately and adding the results together to give the total flux. However, if in fact the medium is flowing, the standard diffusion equations are no longer strictly applicable. In addition, the equation (Bouldin, [1961]) he quotes for diffusion applies only to a constant concentration at the absorbing surface, a condition that is not fulfilled. The problem is usually presented as though it were possible to assign a contribution from diffusion and another from mass flow.

P. H. Nye and J. A. Spiers, (1964) showed that in general this separation cannot be made since the two processes interact. They assumed that :

(a) there is a linear relation between the concentration of substance in solution and its total concentration in soil, over the range of concentration in which we are interested.

(b) the absorbing surface of the root may be approximated by a cylinder whose surface lies near the tips of the root hairs, and for the present argument we neglect the substance within the root hair zone.

(c) the rate of uptake varies as the concentration in the soil solution at this surface.

They were interested in knowing the pattern of accummulation or depletion around, the root that appears at successive intervals of time and how much substance has been taken up by the root element. They solved the problem, i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}(D' r\frac{\partial C}{\partial r}+\gamma_0 r_0 C)=\frac{\partial C'}{\partial t}$$

under the assumption that $\frac{\partial C'}{\partial t}=0$

as they could not discover any general solution to this equation.

They got the solution as

$$C=C_i\left[1+\beta\left(\frac{r}{r_0}\right)^{-\nu}\right]$$

where

$$Y=\frac{\gamma_0 r_0}{D'}$$

and $\beta=\frac{\gamma_0-k}{k}.$

Under these assumptions they noted that $C>C_i$ if $\gamma_0>k$ corresponding to accumulation and $C<C_i$ if $\gamma_0<k$ corresponding to depletion (At the root surface $C_0=C_i\frac{\gamma_0}{k}$). Here, in this paper, we try to give a theoretical solution of the problem under unsteady state but with the assumption that

$$C'=C \text{ and therefore } D'=\frac{dC'}{dC}D=D$$

where D' is the apparent diffusion co-efficient.

2. *Mathematical Statement of the Problem :*

$$\frac{1}{r}\frac{\partial}{\partial r}\left[D\gamma\frac{\partial C}{\partial r}+\gamma_0 r_0\, C\right]=\frac{\partial C}{\partial t} \tag{1}$$

under the conditions

$$C=0\ ,\quad r=a,\quad t=0 \tag{2}$$

$$C=A_1,\quad r=a,\quad t>0 \tag{3}$$

$$C=A_2,\quad r=b,\quad t>0 \tag{4}$$

3. *Solution of the Problem :*

On differentiating (1) reduces to

$$D\frac{\partial^2 C}{\partial r^2}+(D+\gamma+\gamma_0 r_0)\frac{1}{r}\frac{\partial C}{\partial r}=\frac{\partial C}{\partial t} \tag{5}$$

or

$$\frac{\partial^2 C}{\partial r^2}+\frac{A}{D}\frac{1}{r}\frac{\partial C}{\partial r}=\frac{1}{D}\frac{\partial C}{\partial t} \tag{6}$$

where $A=D+\gamma_0 r_0$

Taking L. T. of (6) and using (2)

$$\frac{d^2\overline{C}}{dr^2}+\frac{A}{Dr}\frac{d\overline{C}}{dr}-\frac{S}{D}\overline{C}=0. \tag{7}$$

Let $\frac{A}{D}=n,\quad \frac{S}{D}=q^2=u^2\ (S=\lambda \text{ or } p).$

The general solution of (7) can be written as

$$\overline{C}=r\left[C_1K_\gamma\ (qr)+C_2\ I_\gamma\ (qr)\right] \tag{8}$$

where $\gamma=\frac{D-A}{2D}.$

Making use of the conditions (3) and (4)

$$C_2=\frac{[\phi_1(q)]}{S\phi_2(q)} \tag{9}$$

$$C_1=\frac{1}{S}\left[\frac{\Psi_1(q)}{\phi_2(q)}\right] \tag{10}$$

where $\phi_1(q)=A_1a^{-\gamma}K_\gamma\,(qb)-A_2K_\gamma\,(qa)\ b^{-\gamma}$ (11)

$$\psi_1(q)=A_1a^{-\gamma}\,I_\gamma\,(qb)-A_2\ b^{-\gamma}\,I_\gamma\,(qa) \tag{12}$$

$$\phi_2(q) = I_\gamma(qa)\,K_\gamma(qb) - I_\gamma(qb)K_\gamma(qa). \tag{13}$$

$$\therefore \quad \overline{C} = \frac{r^\gamma}{S}\left[\frac{\phi_1(q)I_\gamma(qr) - \phi_1(q)K_\gamma(qr)}{\phi_2(q)}\right].$$

C is now determined by the Laplace inversion formula. The integral is a single-valued function of λ with a simple pole at $(S)=0$ and simple poles at $\lambda = -Da_n^2$ or $\mu = ia_n$ where $\pm\alpha_n$ are the roots (all real and simple) of $I_\gamma(\alpha a)K_\gamma(\alpha b) - I_\gamma(\alpha b)K_\gamma(\alpha a) = 0.$ (15)

On differentiating $\phi_2(q)$ and evaluating at $\mu = i\alpha_n$,

$$\phi_2(i\alpha_n = a)\left[I'_\gamma(i\alpha_n a)K_\gamma(i\alpha_n b) - I_\gamma(i\alpha_n b)K'_\gamma(i\alpha_n a)\right]$$
$$+ b\left[I_\gamma(i\alpha_n a)K'(i\alpha_n b) - I'_\gamma(i\alpha_n b)K_\gamma(i\alpha_n a)\right]. \tag{16}$$

The residue at the pole at $\lambda = 0$ is obtained by taking the limit $\lambda \to 0$ of $S\overline{C}$. Thus we have it equal to

$$r^\gamma\left[\left(A_1 a^{\gamma}\frac{\Gamma(\gamma)2^{\gamma-1}}{(qb)^\gamma} - \frac{A_2\Gamma(\gamma)2^{\gamma-1}b^{-\gamma}}{(qa)^\gamma}\right)\frac{(qr)^\gamma}{2^\gamma\Gamma(1+\gamma)}\right.$$
$$\left. - \left(A^1 a^{-\gamma}\frac{(qb)^\gamma}{2^\gamma\Gamma(1+\gamma)} - A_2\frac{b^{-\gamma}(qa)^\gamma}{2^\gamma\Gamma(1+\gamma)}\right)\frac{\Gamma(\gamma)2^{\gamma-1}}{(qr)^\gamma}\right]$$

$$\underset{\lambda\to 0}{Lt}\left[\frac{(qa)^\gamma}{2^\gamma\Gamma(1+\gamma)}\;\frac{\Gamma(\gamma)2^{\gamma-1}}{(qb)^\gamma} - \frac{(qb)^\gamma}{2^\gamma\Gamma(1+\gamma)}\;\frac{\Gamma(\gamma)2^{\gamma-1}}{(qa)^\gamma}\right]$$

$$= \frac{\left[(A_1 - A_2)\,r^{2\gamma} - (A_1 b^{2\gamma} - A_2 a^{2\gamma})\right]}{r^\gamma(a^{2\gamma} - b^{2\gamma})}. \tag{17}$$

For calculating the residue at other poles we would find out

$$\left.\frac{d\phi_2}{d\lambda}\right|_{= -D\alpha_n^2}$$
$$= \tfrac{1}{2}\mu\left.\frac{d\phi_2}{d\mu}\right|_{\mu = i\alpha_n} \tag{18}$$
$$= \tfrac{1}{2}\alpha_n\frac{d\phi_2}{d\alpha_n}.$$

Noting that

$$\left[I_\gamma(ia\alpha)K_\gamma(ib\alpha) - K_\gamma(ia\alpha)I_\gamma(ib\alpha)\right]$$
$$= -\frac{\pi}{2}\left[J_\gamma(a\alpha)Y_\gamma(b\alpha) - Y_\gamma(a\alpha)J_\gamma(b\alpha)\right] \tag{19}$$

$$\frac{d}{d\alpha}\Big[J_\gamma(a\alpha)\,Y_\gamma(b\alpha)-Y_\gamma(a\alpha)\,J_\gamma(b\alpha)\Big]$$

$$=aJ'_\gamma(a\alpha)\,Y_\gamma(b\alpha)+bJ_\gamma(a\alpha)\,Y'_\gamma(b\alpha)$$

$$-aY'_\gamma(a\alpha)J_\gamma(b\alpha)-bY_\gamma(a\alpha)J'_\gamma(b\alpha) \tag{20}$$

$$\frac{J_\gamma(a\alpha)}{J_\gamma(b\alpha)}=\frac{Y_\gamma(a\alpha)}{Y_\gamma(b\alpha)}=K \tag{22}$$

Thus (20) becomes

$$=bk\Big[J_\gamma(b\alpha_1)\;Y'_\gamma(b\alpha_1)-J'_\gamma(b\alpha_1)\,Y_\gamma(b\alpha_1)\Big]$$

$$-\frac{a}{k}\Big[J_\gamma(a\alpha_1)\,Y'_\gamma(a\alpha_1)-J'_\gamma(a\alpha_1)\,Y_\gamma(a\alpha_1)\Big]$$

$$=\frac{2}{\pi\alpha_1}\Big(k-\frac{1}{K}\Big)=\frac{2}{\pi\alpha_1}\;\frac{J^2_\gamma(a\alpha_1)-J^2_\gamma(b\alpha_1)}{J_\gamma(a\alpha_1)\,J_\gamma(b\alpha_1)}. \tag{23}$$

Also

$$\Big[\phi_1(q)I_\gamma(qr)-\psi_1(q)\,K_\gamma(qr)\Big]_{\mu=i\alpha_n}$$

$$=A_1a^{-\gamma}\Big[-\frac{\pi}{2}\Big\{J_\gamma(r\alpha)Y_\gamma(b\alpha)-Y_\gamma(r\alpha)J_\gamma(b\alpha)\Big\}\Big]$$

$$+A_2b^{-\gamma}\Big[-\frac{\pi}{2}\Big\{J_\gamma(a\alpha)Y_\gamma(r\alpha)-Y_\gamma(a\alpha)J_\gamma(r\alpha)\Big\}\Big] \tag{24}$$

On using (19)

Thus the residue at $\lambda=-D\alpha_n^2$

$$=e^{-D\alpha_n^2t}\Big[A_1a^{-\gamma}\Big(-\frac{\pi}{2}\Big)\Big\{J_\gamma(r\alpha_n)\,Y_\gamma(b\alpha_n)-Y_\gamma(r\alpha_n)J_\gamma(b\alpha)\Big\}$$

$$\frac{+A_2b^{-\gamma}\Big(-\frac{\pi}{2}\Big)\Big\{J_\gamma(a\alpha_n)\,Y_\gamma(r\alpha_n)-Y_\gamma(a\alpha_n)J_\gamma(r\alpha_n)\Big\}\Big]}{}$$

$$\Big[\frac{1}{2}\alpha_n\frac{d}{d\alpha_n}\{\phi_2\}\Big]_{\mu=i\alpha_n}=(-)\frac{1}{2}\;\frac{J_\gamma^{\,2}(a\alpha_n)-J_\gamma^{\,2}(b\alpha_n)}{J_\gamma(a\alpha_n)\,J_\gamma(b\alpha_n)} \tag{25}$$

care of (18)

or

$$e^{-D\alpha_n^2 t}\frac{\left[A_1 a^{-\gamma}\left(-\frac{\pi}{2}\right)\left\{J_\gamma(r\alpha_n)Y_\gamma(b\alpha_n)-Y_\gamma(r\alpha_n)J_\gamma(b\alpha_n)\right\} + A_2 b^{-\gamma}\left\{\left(-\frac{\pi}{2}\right)J_\gamma(a\alpha_n)Y_\gamma(r\alpha_n)-Y_\gamma(a\alpha_n)J_\gamma(r\alpha_n)\right\}\right]}{\frac{1}{2}\alpha_n\left(-\frac{\pi}{2}\right)\left(\frac{2}{\pi\alpha_n}\right)\frac{{}^2J_\gamma(a\alpha_n)-J^2_\gamma(b\alpha_n)}{J_\gamma(a\alpha_n)J_\gamma(b\alpha_n)}}$$

using (23). (26)

Using the inversion formula

$$C=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{r^\gamma e^{\lambda t}}{\lambda}\left[\frac{\phi_1(q)I_\gamma(qr)-\psi_1(q)K_\gamma(qr)}{\phi_2(q)}\right]d\lambda \tag{27}$$

C = sum of (26) and (17).

The steady-state solution is given by equation (17) only.

List of Nomenclatures used

C' = Concentration of substance in soil.

C = Concentration of substance in the soil solution.

C_0 = Concentration of substance in the soil solution at absorbing surface.

γ_0 = Inward radial velocity at absorbing surface.

C_i = Concentration of substance in the soil solution initially.

r_0 = Radius at absorbing surface.

I_0 = Outward radial flux of substance at the surface.

$k = I_0/C_0$

$\gamma_0 r_0$ = The measure of radial flux of water.

$I_\gamma(x)$ = Modified Bessel Function of the first kind of order.

$K_\gamma(x)$ = Modified Bessel Function of second kind of order.

REFERENCES

1. Bouldin, D. R., 1961, Math. description of diffusion process in soil plant system, Soil Sci. Soc. Amer. Proc. 25, 276-480.

2. Passioura, J. B., 1963, A Math, Model for the uptake of ions from the soil solution, Plant and Soils, 18, 225-238.

3. Nye, P. H. and Pspiers, J. A., 1964, Univ. of Oxford, England, 8th Inter. Cong. of soil Sci. Bucharest, Romania, III, p. 535.