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MHD FLOW WITH HALL EFFECTS,...INDUCED BY TORSIONAL OSCILLATIONS OF A POROUS DISC IN A ROTATING FLUID

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ABSTRACT

The hydromagnetic flow induced by torsional oscillations of a porous disc in a rotating fluid under uniform injection, has been studied in the presence of a transverse magnetic field. Hall effect has been taken into consideration. Assuming the frequency and the amplitude oscillation to be small, the equations of motion have been solved by using the expressions for velocity components and pressure in the exponential form of non-dimensional time. Amplitude and phase of axial velocity at infinity, phase of the transverse shear stress at the disc and boundary layer thicknesses against a non-dimensional parameter, depending on the injection velocity, the angular velocity and the kinematic viscosity have been studied. It has been found that the oscillating axial velocity at large distances from the disc has always a phase lead for large values of this parameter.

1. *Introduction* :

Datta and Jana [1] have studied the Hall effects on the oscillating *MHD* flow past a flat plate in the presence of a uniform transverse magnetic field. The effects of Hall Current on the torsional oscillation of a disc in a conducting fluid subjected to a uniform axial magnetic field has been studied by Datta [2]. Gupta [3] has studied the flow induced by the torsional oscillation of a disc about a state of rotation with angular velocity Ω in a semi-infinite electrically conducting fluid which is also rotating with the same angular velocity.

The objective of the present paper is to investigate the effect of injection on the *MHD* flow induced by torsional oscillations of a porous disc in a rotating fluid in the presence of a transverse magnetic field. Hall effects have been taken into consideration. Considering the amplitude and frequency of oscillation to be small, modified Navier-Stokes equations, Ohm's law and Maxwell's equations have been solved by using the expressions for velocity and pressure in the exponential form of non-dimensional time and neglecting the second order terms. The effects of injection on the flow have been investigated by calculating the results for various values of a non-dimensional parameter λ (depending on the injection velocity U_0, angular velocity of the disc and the kinematic viscosity ν). It has been observed that the axial velocity at large distances from the disc has always a phase lead for large values of λ. Jana and Datta [4] have studied this flow geometry without taking injection into account.

The Physical Problem and its Solution :

We suppose that an infinite porous disc located at $z=0$ executes torsional oscillation with small amplitude ϵ and frequency n^* about a state of steady rotation with angular velocity Ω about z-axis in a conducting fluid which is also rotating with the same angular velocity. Fluid of same density is injected through the disc with uniform velocity $2U_0$. We choose a cylindrical polar co-ordinates system (r, θ, z) and consider the flow to be axisymmetric so that all the physical quantities are functions of r and z. We also assume that the induced magnetic field is negligible in comparison with the imposed uniform magnetic field $\vec{B}_0$ parallel to z-axis, which is justified for flows with small magnetic Reynolds number. In order to consider Hall effects the Ohm's law is modified. The governing equations of the problem, are

$$\text{div}\ \vec{V}=0 \tag{1}$$

$$\frac{\partial \vec{V}}{\partial t}+(\vec{V}.\text{grad})\,\vec{V}=-\frac{1}{\rho}\ \text{grad}\ p+\nu+\Delta^2\ \vec{V}\,\frac{1}{\rho}\,\vec{J}\times\vec{B} \tag{2}$$

$$\vec{J}=\sigma(\vec{E}+\vec{V}\times\vec{B})-\frac{\sigma}{en_e}\,\vec{J}\times\vec{B} \tag{3}$$

$$\text{curl}\ \vec{B}=\mu\,\vec{J},\ \text{curl}\ \vec{E}=0,\ \text{div}\ \vec{B}=0 \tag{4}$$

Where $\vec{V}$ is the velocity vector, t the time, $\vec{B}$ the magnetic induction vector, $\vec{E}$ the electric intensity vector, $\vec{J}=(J_r, J_\theta, J_z)$ the electric current density vector, p the fluid pressure, ρ the density of the fluid. υ the kinematic viscosity, σ the electrical conductivity, e the electric charge, n_e the number density of electrons and μ is the magnetic permeability. On writing (3), the ion-slip and the thermoelectric effects are neglected and further it is assumed that $\omega_i \tau_i \ll 1$ where ω_i and τ are the cyclotron frequency and the collision time of ions respectively. For partially ionised gases, the electron pressure gradient may be neglected.

Under the above assumptions the basic equations (1)—(4) lead to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right]$$

$$+\frac{\sigma B_0{}^2}{\rho(1+m^2)}[m(v-r\Omega)-u] \tag{5}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=v\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 z}{\partial z^2}\right]$$

$$-\frac{\sigma B_0{}^2}{\rho(1+m^2)}[(v-r\Omega)+mu] \tag{6}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right] \tag{7}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{8}$$

Here u, v, w are the velocity components of fluid, $m=\omega_e \tau_e$ where ω_e and τ_e are respectively the cyclotron frequency of electrons and the collision time of electrons with ions.

Boundary conditions are

$$u=0,\ v=r\Omega+r\epsilon n^*\cos n^*t,\ w=2U_0 \quad \text{at } z=0$$
$$u\to 0,\ v\to r\Omega \quad \text{as } z\to\infty \tag{9}$$

We seek a solution of (5)—(8) for u, v and w in the following form :

$$u = rf(z)\ \text{Exp}\ (in^*t)$$
$$v = r\Omega+rg(z)\ \text{Exp}\ (in^*t)$$

$$w = 2U_0 + h(z)\ \text{Exp}\ (in^*t)$$

$$p = p_0 + \tfrac{1}{2}\rho r^2\Omega^2 + p_1(r,z)\ \text{Exp}\ (in^*t)$$

Where p_0 is a constant. Substituting (10) in (5)—(8) and neglecting 2nd order terms, we get,

$$(in^*) f + 2U_0 f' - 2\Omega g = -\frac{1}{r\rho}\frac{\partial p_1}{\partial r} + \nu f'' + \frac{\Omega N^2}{1+m^2}(mg - f) \tag{11a}$$

$$(in^*) g + 2\Omega f + 2U_0 g' = \nu g'' - \frac{\Omega N^2}{1+m^2}(g + mf) \tag{11b}$$

$$(in^*) h + 2U_0 h' = -\frac{1}{\rho}\frac{\partial p_1}{\partial z} + \nu h'' \tag{11c}$$

$$2f + h' = 0 \tag{11d}$$

Where $N^2 = \sigma B_0{}^2/(\rho\Omega)$

The boundary conditions (9) become,

$$f = 0,\ g = \epsilon n^*,\ h = 0 \qquad \text{at } z = 0$$

$$f \to 0,\ g \to 0 \qquad \text{as } z \to \infty \tag{12}$$

Differentiating equation (11c) with respect to r we get $\frac{\partial p_1}{\partial r \partial z} = 0$, which implies that $\frac{\partial p_1}{\partial r}$ is a constant, so that from equation (11a) with the condition that $f \to 0$ and $g \to 0$ as $z \to \infty$, we get $\frac{\partial p_1}{\partial r} = 0$. The equations (11) then reduce to

$$(in^*) f + 2U_0 f' - 2\Omega g = \nu f'' + \frac{\Omega N^2}{1+m^2}(mg - f) \tag{13}$$

$$(in^*) g + 2U_0 g' + 2\Omega f = \nu g'' - \frac{\Omega N^2}{1+m^2}(g + mf) \tag{14}$$

$$2f + h' = 0 \tag{15}$$

To solve equations (13)—(14) we consider two functions F and $\overline{F}$ such that

$$F = f + ig \text{ and } \overline{F} = f - ig \tag{16}$$

the equations (13) and (14) then become

$$\nu F'' - 2U_0 F' - [in^* + 2i\Omega + \frac{\Omega N^2}{1+m^2}(im+1)]F = 0 \tag{17}$$

$$\nu \bar{F}'' - 2U_0 \bar{F}' + [-in^* + 2i\Omega + \frac{\Omega N^2}{1+m^2}(im-1)]\bar{F} = 0 \tag{18}$$

The boundary conditions (12) become

$$F = i\epsilon n^*, \quad \bar{F} = -i\epsilon n^* \quad \text{at} \quad z=0$$

$$F \to 0 \;, \quad \bar{F} \to 0 \qquad \text{as} \quad z \to \infty \tag{19}$$

The solutions of (17) and (18) satisfying (19) are

$$F = i\epsilon n^* \, Exp\,[-\{(\alpha_1 - \lambda) + i\beta_1\} \sqrt{\Omega/\nu}\, z] \tag{20}$$

$$\bar{F} = -i\epsilon n^* \, Exp\,[-\{(\alpha_2 - \lambda) + i\beta_2\} \sqrt{\Omega/\nu}\, z] \tag{21}$$

where

$$\alpha_1 = \frac{1}{\sqrt{2}}[\sqrt{(\lambda^2+a)^2+b^2} + (\lambda^2+a)]^{\frac{1}{2}}$$

$$\beta_1 = \frac{1}{\sqrt{2}}[\sqrt{(\lambda^2+a)^2+b^2} - (\lambda^2+a)]^{\frac{1}{2}}$$

$$\alpha_2 = \frac{1}{\sqrt{2}}[\sqrt{\lambda^2+a)^2+b_3{}^2} + (\lambda^2+a)]^{\frac{1}{2}}$$

$$\beta_2 = \frac{1}{\sqrt{2}}[\sqrt{(\lambda^2+a)^2+b_3{}^2} - (\lambda^2+a)]^{\frac{1}{2}} \tag{22}$$

$$\lambda = U_0/\sqrt{\nu\Omega}, \; a = N^2/(1+m^2), \; b = n+P, \; n = n^*/\Omega$$

$$P = 2 + \frac{mN^2}{1+m^2}, \; b_3 = n - P$$

Introducing

$$\eta = \sqrt{\Omega/\nu}\; z \tag{23}$$

and using definitions for F and $\bar{F}$, we get

$$f = \tfrac{1}{2} i\epsilon n^* [Exp\{-((\alpha_1 - \lambda) + i\beta_1)\eta\} - Exp\{-((\alpha_2 - \lambda) + i\beta_2)\eta\}] \tag{24}$$

$$9 = \tfrac{1}{2}\epsilon n^* [Exp\{-((\alpha_1 - \lambda) + i\beta_1)\eta\} + Exp\{-((\alpha_2 - \lambda) + i\beta_2)\eta\}] \tag{25}$$

From (15), using (24), we get the expression for $h(\eta)$ as

$$h(\eta)=i\epsilon n^*(\nu/\Omega)^{1/2}\Big[\Big(\frac{Exp\{-((\alpha_1-\lambda)+i\beta_1)\eta\}}{(\alpha_1-\lambda)+i\beta_1}-\frac{Exp\{-((\alpha_2-\lambda)+i\beta_2)\eta\}}{(\alpha_2-\lambda)+i\beta_2)}\Big)$$

$$+\Big(\frac{1}{(\alpha_2-\lambda)+i\beta_2}-\frac{1}{(\alpha_1-\lambda)+i\beta_1}\Big)\Big] \qquad (26)$$

Substituting (24)—(26) in (10) we get the radial, a azimuthal and axial velocities in real form as,

$$u=\frac{r\epsilon n^*}{2}[\sin(\beta_1\eta-n\tau)\,Exp\{-(\alpha_1-\lambda)\eta\}-\sin(\beta_2\eta-n\tau)Exp$$

$$\{-(\alpha_2-\lambda)\eta\}] \qquad (27)$$

$$v-r\Omega=\frac{r\epsilon n^*}{2}[\cos(\beta_1\eta-n\tau)\,Exp\{-(\alpha_1-\lambda)\eta\}$$

$$+\cos(\beta_2\eta-n\tau)\,Exp\{-(\alpha_2-\lambda)\eta\}] \qquad (28)$$

$$w-2u_0=\epsilon n^*(\nu/\Omega)^{1/2}\frac{Exp(-(\alpha_1-\lambda)\eta)}{(\alpha_1-\lambda)^2+\beta_1^2}[\beta_1\cos(n\tau-\beta_1\eta)$$

$$-(\alpha_1-\lambda)\sin(n\tau-\beta_1\eta)]-\epsilon n^*(\nu/\Omega)^{1/2}\frac{Exp(-(\alpha_2-\lambda)\eta)}{(\alpha_2-\lambda)^2+\beta_2^2}$$

$$[\beta_2\cos(n\tau-\beta_2\eta)-(\alpha_2-\lambda)\sin(n\tau-\beta_2\eta)]$$

$$+\epsilon n^*(\nu/\Omega)^{1/2}R\sin(n\tau-\gamma) \qquad (29)$$

where

$$R^2=\frac{(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2}{\{(\alpha_1-\lambda)(\alpha_2-\lambda)+(\beta_1\beta_2\}^2+\{\beta_2(\alpha_1-\lambda)-\beta_1(\alpha_2-\lambda)\}^2}$$

$$\gamma=\tan^{-1}\frac{\beta_1\beta_2(\beta_2-\beta_1)+\{\beta_1(\alpha_2-\lambda)^2-\beta_2(\alpha_1-\lambda)^2\}}{(\alpha_1-\lambda)(\alpha_2-\lambda)(\alpha_2-\alpha_1)+\{\beta_2{}^2(\alpha_1-\lambda-\beta_1{}^2(\alpha_2-\lambda)\}} \qquad (30)$$

3. *Results and Discussions* :

The thicknesses of the boundary layers are given by $1/(\alpha_1-\lambda)$ and $1/(\alpha_2-\lambda)$. From expressions (22) for α_1 and α_2, it is observed that both the boundary layer thicknesses decrease with increase in N^2; on the

Table 1 : Boundary layer thickness $1/(\alpha_1-\lambda)$ against λ for $N^2=12$.

	$m=1$				$m=3$			
λ/n	0.5	1.5	0.6	0.10	0.5	1.5	0.6	0.10
0.5	0.4170	0.4051	0.3591	0.3271	0.6827	0.6308	0.4847	0.4123
1.5	0.5989	0.5789	0.5000	0.4454	1.2711	1.1343	0.7761	0.6186
2.5	0.8438	0.8163	0.7005	0.6156	2.3684	2.0904	1.3183	0.9833
3.5	1.1354	1.1038	0.9601	0.8445	3.9026	3.5035	2.2203	1.5975
4.5	1.4524	1.4202	1.2634	1.1251	5.6466	5.1870	3.4812	2.5151

Table 2 : Boundary layer thicknesses $1/(\alpha_2-\lambda)$ against λ for $N^2=12$.

	$m=1$				$m=3$			
λ/n	0.5	1.5	0.6	0.10	0.5	1.5	0.6	0.10
0.5	0.4293	0.4419	0.4924	0.4924	0.7477	0.8314	1.3979	0.8039
1.5	0.6194	0.6402	0.7179	0.7179	1.4457	1.6705	2.7739	1.5970
2.5	0.8712	0.8979	0.9886	0.9886	2.6998	3.0833	4.3393	2.9634
3.5	1.1658	1.1944	1.2847	1.2847	4.3369	4.7883	5.9584	4.6527
4.5	1.4826	1.5105	1.5944	1.5944	6.1133	6.5648	7.5978	6.4325

Table 3 : Phase (ϕ) of the transverse shear-str ss against λ for small n.

n	$m=1$				$m=3$			
	$N^2=6$		$N^2=8$		$N^2=6$		$N^2=8$	
λ/n	0.5	1.5	0.5	1.5	0.5	1.5	0.5	1.5
0.5	0.6148	0.6073	0.5776	0.5726	0.8697	0.8639	0.8445	0.8397
1.5	0.7843	0.7758	0.7280	0.7223	1.1372	1.1232	1.0962	1.0861
2.5	0.8895	0.8824	0.8286	0.8236	1.2787	1.2647	1.2401	1.2292
3.5	0.9452	0.9403	0.8863	0.8826	1.3395	1.3303	1.3063	1.2986
4.5	0.9749	0.9716	0.9187	0.9161	1.3679	1.3618	1.3380	1.3329

Table 4 : Phase (ϕ) of the transverse shear-stress against λ for large n.

	$m=1$				$m=3$			
	$N^2=6$		$N^2=8$		$N^2=6$		$N^2=8$	
λ/n	0.6	0.10	0.6	0.10	0.6	0.10	0.6	0.10
0.5	0.5769	0.7125	0.4853	0.6460	0.8589	0.8620	0.8240	0.8490
1.5	0.7451	0.8768	0.6341	0.7961	1.0804	1.0618	1.0344	1.0447
2.5	0.8720	1.0038	0.7544	0.9144	1.2289	1.2157	1.1817	1.1938
3.5	0.9551	1.0925	0.8373	0.9996	1.3210	1.3185	1.2768	1.2946
4.5	1.0053	1.1504	0.8902	1.0568	1.3737	1.3821	1.3326	1.3584

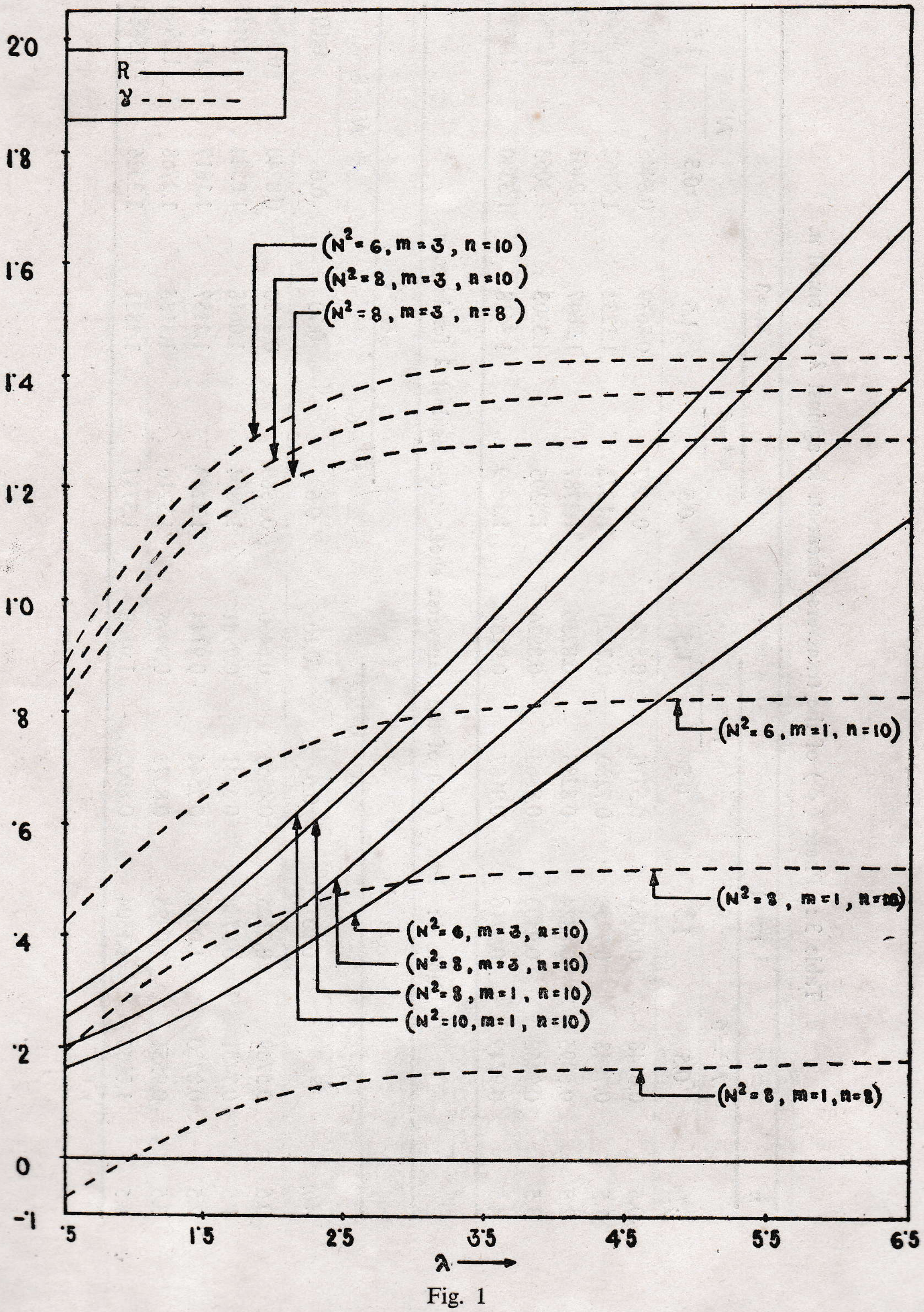

Fig. 1

Amplitude (R) and phase of the (δ) the axial velocity against λ

other hand, both of them increase with the increase of λ. Values of the boundary layer thicknesses $1/(\alpha_1-\lambda)$ and $1/(\alpha_2-\lambda)$ are given respectively in the tables 1 and 2. It is clear from the tables that for any value of n, their rate of increments with the increase of λ are large and small according as the Hall parameter m is large and small respectively. Also boundary layer thicknesses increase with the increase of m when other parameters are kept constant.

From equation (29) the axial velocity at infinity is given by

$$w(\alpha, \tau)-2U_0=\epsilon n^*(\nu/\Omega)^{\frac{1}{2}}R\sin(n\tau-\lambda) \tag{31}$$

Where R and γ are given by equations (30).

The values of R and γ have been plotted against λ for different values of N^2, m and n in figure 1. It is observed that values of R increase with λ for all m, N^2 and n. With the increase of λ, the values of γ gradually increase, reach maximum values and then remain constant throughout for all other values of λ. It is noted from the figure that the oscillating axial velocity at infinity has always a phase lead for large values of λ.

The transverse shear stresses at the disc is given by

$$\rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = -A\cos(n\tau+\phi) \tag{32}$$

Where,

$$\phi=\tan^{-1}\frac{\beta_1+\beta_2}{(\alpha_1-\lambda)+(\alpha_2-\lambda)}$$

and

$$A^2=\frac{\rho^2\nu^2r^2\epsilon^2n^*}{4}\left(\frac{\Omega}{\nu}\right)[\{(\alpha_1-\lambda)+(\alpha_2-\lambda)\}^2+\{\beta_1+\beta_2\}^2]$$

The values of ϕ have been given against λ for different values of N^2, m and n in tables 3 and 4. It is seen that ϕ is much affected by the small values of λ. As λ increases ϕ tends to become constant so that ϕ is not affected by large values of λ.

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