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BOUNDARY LAYER FLOW WITH POWER-LAW VARIATION OF THE FREE STREAM VELOCITY

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ABSTRACT

A new variation method based on the governing principle of dissipative processes is developed to obtain an approximate analytical description of the boundary layer flow for a power-law variation of the external stream velocity. The principle is formulated for the boundary layer equations and a third order profile is assumed for the longitudinal velocity inside the boundary layer region. The skin friction at the wall is found to be quite close to the numerical values.

Introduction :

The governing principle of dissipative processes which describes the evolution of linear, quasi-linear and non-linear irreversible processes was formulated by Gyarmati (1969). The principle in its most general form is written as

$$\delta \int_V (\sigma - \psi - \phi)\, dV = 0 \tag{1}$$

where V is the volume of the system. σ denotes the entropy production which is expressed as a bilinear function of thermodynamic forces $\underline{X}_i$ and the conjugated fluxes $\underline{J}_i$, *i.e.*,

$$\sigma = \sum_{i=1}^{f} \underline{J}_i \cdot \underline{X}_i \geqslant 0. \tag{2}$$

According to the linear Onsager theory the fluxes and forces are related by linear constitutive relations

$$\underline{J}_i = \sum_{k=1}^{f} L_{ik}\, \underline{X}_{ki}$$

or

$$\underline{X}_i = \sum_{k=1}^{f} R_{ik}\, \underline{J}_k \tag{3}$$

where the coefficients L_{ik} and R_{ik} are the conductivities and resistances respectively, the matrices of which are mutually reciprocal and symmetric (1931, a, b). The local dissipation potentials ψ and ϕ are defined as the homogeneous quadratic forms of thermodynamic forces and fluxes respectively, *i.e.*,

$$\psi(\underline{X}, \underline{X}) \equiv \frac{1}{2} \sum_{i,k=1}^{f} L_{ik}\, \underline{X}_i \cdot \underline{X}_k \geqslant 0$$

$$\phi(\underline{J}, \underline{J}) \equiv \frac{1}{2} \sum_{i,k=1}^{f} R_{ik}\, \underline{J}_i \cdot \underline{J}_k \geqslant 0 \tag{4}$$

which correspond to the entropy production (2). Using (2), (3) and (4), the principle (1) becomes

$$\delta \int_V \left[\sum_{i=1}^{f} \underline{J}_i \cdot \underline{X}_i - \frac{1}{2} \sum_{i,k=1}^{f} L_{ik}\, \underline{X}_i \cdot \underline{X}_k - \frac{1}{2} \sum_{i,k=1}^{f} R_{ik}\, \underline{J}_i \cdot \underline{J}_k \right] dV = 0. \tag{5}$$

This principle has already been used to derive the governing equation of fluid flow and heat transfer by Vincze (1971), while the use of the principle to get the solution of Benard convection in hydrodynamic stability was made by Singh (1976). The aim of the present investigation is to see the applicability of this genuine variational formulation of irreversible processes to boundary layer flow. The principle is formulated for the boundary layer along a solid surface when the free stream velocity varies as a power function of $x(U=cx^m)$ where x measures the distance along the surface of the body. A third order trial function is chosen for the longitudinal velocity component which has boundary layer thickness as a variational parameter. The Euler-Lagrange equation associated to the principle gives an algebraic equation in terms of boundary layer thickness which can be solved easily for any value of m. The skin friction obtained by the present method is very close to the numerical values.

Boundary Layer Equations and the Actual Form of the Principle :

Interest in the theory of boundary layer flows is due to numerous engineering problems it occurs in. According to this theory the irreversible process of momentum transfer in flows around bodies occurs mainly inside a very thin layer next to the wall. Therefore, the natural way to study this non-equilibrium process is by using the method of irreversible thermodynamics.

After applying the usual boundary layer approximations, the conservation equations of mass and momentum for two dimensional steady flows reduce to [see Schlichting (1968)].

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

Here the fluid is considered to be incompressible. u and v are the velocity components inside boundary layer along x and y directions respectively where x measures the distance along the body and y is normal to the wall. U denotes the free stream velocity and is assumed of the form

$$U=cx^m. \tag{8}$$

The boundary conditions of the problem are

$$\begin{aligned} y=0 &: \quad u=0, \quad v=0, \\ y\to\infty &: \quad u\to U. \end{aligned} \tag{9}$$

In the formulation of Gyarmati's principle, the balance equations play the basic role which in this case are

$$\nabla\cdot \underline{v}=0 \tag{10}$$

$$(\underline{v}\cdot\nabla)\,\underline{v}+\nabla\cdot \underline{\underline{P}}=0 \tag{11}$$

where (10) represents the mass balance and (11) is the momentum balance. $\underline{\underline{P}}$ is the pressure tensor and is written as

$$\underline{\underline{P}}=p\delta+\underline{\underline{\overset{0}{P}}}{}^{vs}$$

Here p denotes the hydrostatic pressure and $\underline{\underline{\overset{0}{P}}}{}^{vs}$ is the viscous part of the pressure tensor and its trace is zero. $\underline{v}$ is the velocity vector

$$v=ui+vj.$$

In the case of viscous fluid flow, the energy picture of the principle is preferable to that of entropy picture, we therefore use the energy dissipation $T\sigma$ instead of entropy production σ. The energy dissipation in this case is [Gyarmati (1970)]

$$T\sigma=-\underline{\underline{\overset{0}{P}}}{}^{vs} : (\overset{0}{\nabla}\underline{v})^s=-P_{12}\frac{\partial u}{\partial y}\geqslant 0, \tag{12}$$

where $(\overset{0}{\nabla}\underline{v})^s_{12}$ is the symmetric part of the gradient of the velocity with zero trace and in this case it has only one component

$$(\overset{0}{\nabla}\underline{v})^s_{12}=\frac{\partial u}{\partial y}.$$

P_{12} denotes the non-zero component of the viscous pressure tensor $\overset{0}{\underline{\underline{P}}}{}^{vs}$. The double dots represent the scalar product of two tensorial quantities. The constitutive equation in this case are

$$P_{12}=-\mu\frac{\partial u}{\partial y}, \quad (\overset{0}{\nabla}\underline{v})^{s}{}_{12}=-\frac{1}{\mu}P_{12}, \tag{13}$$

where μ denotes the coefficient of viscosity. $(\overset{0}{\nabla}\underline{v})^{s}{}_{12}$ and P_{12} are the thermodynamic force and current respectively. The dissipation potentials ψ and ϕ in energy picture are

$$\psi^{*}=T\psi=\frac{\mu}{2}(\nabla\underline{v})^{2}{}_{12},$$

$$\phi^{*}=T\phi=\frac{1}{2\mu}P^{2}{}_{12}. \tag{14}$$

Using (12) and (14) in (1), the actual form of the principle for the problem under consideration becomes

$$\delta\int_0^L\int_0^d\left[-P_{12}\frac{\partial u}{\partial y}-\frac{\mu}{2}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{2\mu}P^2{}_{12}\right]dy\ dx=0, \tag{15}$$

where L denotes the characteristic length of the surface and d is the viscous boundary layer thickness. This principle with the balance equations (10) and (12) describes the steady two dimensional boundary layer flow along a rigid body. To solve the equations describing the boundary layer flow, we assume the thermodynamic current P_{12} in term of an approximate constitutive relation [Singh (1976)]

$$P_{12}=-\mu\frac{\partial u^*}{\partial y}, \tag{16}$$

where u^* is an approximate velocity component and satisfies the same conditions as u. In exact theory $u=u^*$ and the Lagrangian density in (15) is zero. To get the approximate variational solution of (10) and (11) we introduce (16) in (11) and (15) to get

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u^*}{\partial y^2}, \tag{17}$$

$$\delta\int_0^L\int_0^d\left[\frac{\partial u^*}{\partial y}\frac{\partial u}{\partial y}-\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{2}\left(\frac{\partial u^*}{\partial u}\right)^2\right]dy\ dx=0. \tag{18}$$

Using the similarity transformations

$$u=Uf'(\eta),\ u^*=Uf^{*\prime}(\eta),$$

$$\eta=\left(\frac{1+m}{2}\frac{c}{\nu}x^{m-1}\right)^{1/2}y,$$

$$v=-\left(\frac{2}{1+m}\nu c\ x^{m-1}\right)^{1/2}\left(\frac{m+1}{2}f+\frac{m-1}{2}\eta f'\right) \tag{19}$$

the equation (17) and the principle (18) result to

$$f^{*'''}+ff''+\beta(1-f'^2)=0, \tag{20}$$

$$\delta\int_0^L\int_0^d \left[f^{*''}f''-\frac{1}{2}f''^2-\frac{1}{2}f^{*''2}\right]x^{\frac{5m-1}{2}}\,d\eta\,dx=0, \tag{21}$$

where $\beta=\frac{2m}{1+m}$. We assume the following polynomial for velocity profile

$$\frac{u}{U}=f'=\frac{3}{2}N-\frac{1}{2}N^3+\beta d^2\left(\frac{1}{4}N-\frac{1}{2}N^2+\frac{1}{4}N^3\right) \tag{22}$$

which satisfies the conditions

$$\eta=0: f=0,\ f'=0,\ f'''=-\beta$$
$$\eta\to d: f'\to 1,\ f''\to 0 \tag{23}$$

In (22), d is the variational parameter which is determined with the help of principle (23) and $N=\eta/d$. Using (22) in (20). we get $f^{*''}$ as

$$\begin{aligned}f^{*''}=&d\left(\frac{39}{280}-\frac{3}{8}N^3+\frac{21}{80}N^5-\frac{3}{112}N^7\right)+\beta d\left(\frac{18}{35}-N+\frac{3}{4}N^3-\frac{3}{10}N^5+\frac{N^7}{28}\right)\\&-\beta d^3\left(\frac{1}{560}+\frac{N^3}{8}-\frac{N^4}{4}+\frac{7}{80}N^5+\frac{N^6}{16}-\frac{3}{112}N^7\right)\\&-\beta^2d^3\left(\frac{19}{840}-\frac{N^3}{4}+\frac{3}{8}N^4-\frac{N^5}{10}-\frac{N^6}{12}-\frac{N^7}{28}\right)\\&-\beta^2d^5\left(\frac{1}{1680}+\frac{N^3}{96}-\frac{N^4}{24}+\frac{53}{960}N^5-\frac{N^6}{32}+\frac{3N^7}{448}\right)\\&-\beta^3d^5\left(\frac{1}{1680}-\frac{N^3}{48}+\frac{N^4}{16}-\frac{3N^5}{40}+\frac{N^6}{24}-\frac{N^7}{112}\right),\end{aligned} \tag{24}$$

which satisfies the condition $f^{*''}(d)=0$ since $P_{12}=0$ at the edge of the boundary layer. Substituting $f^{*''}$ and f'' from (22) and (24) into (21) and integrating with respect to η, we get

$$\begin{aligned}\delta\int_0^L\Big[&\frac{600}{d}-(107.811+190.624\beta)d+(4.904+20.629\beta+29.872\beta^2)d^3\\&-(0.295\beta+1.333\beta^2-2.065\beta^3)d^5-(0.041\beta^2+0.079\beta^3\\&-0.045\beta^4)d^7+(0.002\beta^3+0.007\beta^4+0.0046\beta^5)d^9+(0.868\beta^4\\&+0.74\beta^5+1.021\beta^6)10^{-4}d^{11}\Big]x^{\frac{5m-1}{2}}\,dx=0.\end{aligned} \tag{25}$$

The Euler-Lagrange equation of (25) is

$$600+(107.811+190.624\beta)d^2-(14.712+61.887\beta+89.616\beta^2)d^4$$
$$+(1.475\beta+6.665\beta^2+10.325\beta^3)d^6+(0.287\beta^2+0.553\beta^3$$
$$-0.315\beta^4)d^8-(0.018\beta^3+0.063\beta^4+0.0414\beta^5)d^{10}$$
$$-(9.548\beta^4+8.14\beta^5+11.231\beta^6)10^{-3}d^{12}=0. \qquad (26)$$

This is a very general analytical expression from which boundary layer thickness for various values of β or m can be obtained. Thus the important physical characteristics of the boundary layer flow can be studied now. The solution of equation (26) for $\beta=0$, 0.5 and 1 are obtained as $d=3.321$, 2.802 and 2.426 respectively. The most important physical quantity of boundary layer flow is the skin friction at the wall. The non-dimensional skin friction, S^* defined as

$$S^*=\left(\frac{\partial u}{\partial y}\right)y=0/U\sqrt{\frac{U}{\upsilon x}}$$

is calculated for $\beta=0$, 0.5 and 1 and the present result is compared with that of exact one in Table I.

TABLE I

β	*Exact* S^*	*Present* S^*
0	0.332	0.319
0.5	0.755	0.723
1	1.234	1.225

It is clear from the table that the present method is quite satisfactory for boundary layer flows. The result differ from exact solution by less than 4 per cent which is quite satisfactory for engineering applications.

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