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RELIABILITY ESTIMATES FOR TWO PARAMETER EXPONENTIAL DISTRIBUTION*

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ABSTRACT

In this paper we derive the minimum variance unbiased (MVU) estimates of $R=Pr.(X\geqslant Y)$, using Rao-Blackwell and Lehmann-Scheffe theorems, where X and Y are two-parameter exponential strength and stress random variables respectively. The two parameters are minimum stress (strength) parameter and reciprocal of mean. We have considered censored samples of X and Y for the purpose. Three cases are considered viz., when (1) only minimum stress and strength are known, (2) only reciprocal of means are known and (3) all parameters are unknown.

The maximum likelihood estimators of R are also obtained for the above three cases.

Introduction :

Let X and Y be two random variables representing strength of a component and stress working on it, respectively. Then reliability of the component is defined as

$$R=Pr.(X\geqslant Y) \tag{1}$$

In this paper we have considered the problem of estimation of R when X and Y have two parameter exponential distributions given respectively by

$$p(x\,;\,A,\theta)=\frac{1}{\theta}exp\{-(x-A)/\theta\} \tag{2}$$

$$q(y\,;\,B,\mu)=\frac{1}{\mu}exp\{-(y-B)/\mu\} \tag{3}$$

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where A and B represent minimum admissible strength and stress respectively ; and θ and μ represent reciprocal of means of strength and stress respectively. We have found MVU estimators of R from censored samples of X and Y, using Rao-Blackwell and Lehmann-Scheff'e theorems (4). We have considered three cases : (1) When the parameters A and B are known but means of the distributions are unknown, (2) when reciprocal of means i.e., θ and μ are known but A and B are unknown and (3) when all the parameters are unknown.

The estimators of R are also obtained using maximum likelihood methods in the above cases.

Basu (1) used Rao-Blackwell and Lehmann-Scheff'e theorems to derive the estimates of reliability for life time distributions from censored sample. Beg (2) obtained the MVUE for censored samples from truncated life time distributions. For stress-strength model, Tong (5-8) obtained MVU estimates of R for one parameter exponential, Gamma and Exponential family of distributions; but he has considered it for complete samples not for censored samples. The results of Tong (5) and (6) [equation 5 and 3] can be obtained as a special case of our model when $A=B=O$ and samples are complete.

Notations :

X : strength of the component, a r.v.

Y : stress on the component, another r.v.

n and m : complete sample sizes on X and Y,

$x_{(1)} \leqslant x_{(2)} \leqslant \ldots\ldots \leqslant x_{(n)}$: ordered sample from X,

$y_{(1)} \leqslant y_{(2)} \leqslant \ldots\ldots \leqslant y_{(m)}$: ordered sample from Y,

p and v : sizes of ordered samples from X and Y which are available, *i.e.*, sizes of censored samples.

The samples on X and Y are independent. Following Epstein and Sobel [3], let us define some quantities for sample on strength, as

$$W_i=(n-i+1)\,(x_{(i)}-x_{(i-1)})$$

$$U_x=\sum_{i=1}^{r}(x_{(i)}-A)+(n-r)\,(x_{(r)}-A)$$

$$z_x=\sum_{i=1}^{r}(x_{(i)}-x_{(1)})+(n-r)\,(x_{(r)}-x_{(1)})=\sum_{i=2}^{r}w_i$$

Similarly w'_j, $j=1, 2, \ldots\ldots v$, U_y and z_y may be defined for stress.

Now, when X and Y are distributed as (2) and (3) then

$$R=\frac{\theta}{\theta+\mu}exp\{-(B-A)/\theta\}. \tag{4}$$

$A<\theta,\ B<\mu.$

The MVU estimator, $\hat{R}$, of R is obtained as

$$\hat{R}=\int_0^\infty g\left(\xi_y \mid \hat{B},\ \hat{\mu}\right)\left[\int_{\xi_y}^\infty f\left(\xi_x \mid \hat{A},\ \hat{\theta}\right)d\xi_x\right]d\xi_y. \tag{5}$$

or

$$\hat{R}=1-\int_0^\infty f\left(\xi_x \mid \hat{A},\ \hat{\theta}\right)\left[\int_{\xi_x}^\infty g\left(\xi_y \mid \hat{B},\ \hat{\mu}\right)d\xi_y\right]d\xi_x \tag{5'}$$

Where ξ_x and ξ_y are any one of the observations x_i's, $i=1, 2\ldots\ldots, r$ and y_j's, $j=1, 2, \ldots\ldots, v$ from (2) and (3) respectively and $\hat{A}$, $\hat{B}$, $\hat{\theta}$, and $\hat{\mu}$ are the complete sufficient statistics for A, B, θ and μ respectively. If any of these are known, the parameter is taken instead of its estimate in (5) or (5').

Minimum Variance Unbiased Estimation :

Epstein and Sobel [3] have shown that w_i's and w'_j's $i=1, 2,\ldots\ldots, r$ and $j=1, 2,\ldots\ldots, v$, are mutually independent with common p.d.fs. (2) and (3) respectively.

Here for all the following three cases we consider that we have two independent random samples $(w_1, w_2,\ldots\ldots w_i\ldots\ldots w_r)$, $(w'_1, w'_2\ldots\ldots, w'_j,\ldots\ldots, w'_v)$ from (2) and (3) respectively and we find the appropriate conditional distributions to obtain the MVU estimates of R.

(*i*) If A and B are known, U_x and U_y are complete sufficient estimators for θ and μ, respectively.

(*ii*) If θ and μ are known, $x_{(1)}$ and $y_{(1)}$ are complete sufficient estimators for A and B, respectively.

(*iii*) If (A, θ) and (B, μ) are unknown, $(x_{(1)}, z_x)$ and $(y_{(1)}, z_y)$ are complete sufficient estimators for (A, θ) and (B, μ) respectively.

Here we shall consider three cases in the following sections. In Section 1 we assume that A and B are known and in section 2, θ and

μ are known. In section 3 we take all A, B, θ and μ are unknown. In section 4 we consider the m.l.e. of R for all the above three cases.

1. *A and B are known θ and μ are unknown :*

$$\text{Here, } \hat{R}=\int_0^\infty \left(g(\xi_y \mid B, \hat{\mu})\left[\int_{\xi_y}^\infty f\left(\xi_x \mid A, \hat{\theta}\right)\xi_x\right] d\xi_y. \tag{5''}$$

Since A and B are known and U_x and U_y are the respective complete sufficient estimators of θ and μ therefore we shall consider the conditional distributions $f(\xi_x \mid U_x)$ and $g(\xi_y \mid U_y)$ as :

$$f(\xi_x \mid U_x)=(r-1)U_x^{-1}\left[1-\frac{\xi_x-A}{U_x}\right]^{r-2}, \quad A\leqslant\xi_x\leqslant A+U_x \tag{6}$$

$$g(\xi_y \mid U_y)=(v-1)U_y^{-1}\left[1-\frac{\xi_y-B}{U_y}\right]^{v-2}, \quad A\leqslant\xi_y\leqslant B+U_y. \tag{7}$$

Now here may arise four situations viz.

(*i*) $A<B<B+U_y<A+U_x$

(*ii*) $B<A<A+U_x<B+U_y$

(*iii*) $A<B<A+U_x<B+U_y$

(*iv*) $B<A<B+U_y<A+U_x$

In the following subsections we shall consider all the cases.

1.1 $A<B<B+U_y<A+U_x$:

From (5″), (6) and (7) we have

$$\hat{R}=(v-1)\, U_y^{-1}\left(1+\frac{A}{U_x}\right)^{r-1} \sum_{i=0}^{r-1}(-1)^i\frac{\binom{r-1}{i}}{(A+U_x)^i}$$

$$\int_B^{B+U_y}\left(1-\frac{\xi_y-B}{U_y}\right)^{v-2}\xi_y^i\, d\xi_y \tag{8}$$

$$=(v-1)\, U_y^{-1}\left(1+\frac{A}{U_x}\right)^{r-1}\left(1+\frac{B}{U_y}\right)^{v-2}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{v-2}(-1)^{i+j}\frac{\binom{r-1}{i}\binom{v-2}{j}}{(A+U_x)^i(B+U_y)^j}\cdot\frac{\{(B+U_y)^{i+j+1}-B^{i+j+1}\}}{(i+j+1)} \tag{9}$$

Tong's [5] result (eq. 5) can be obtained putting in eq. (8) $A=B=0$, $r=n$, $v=m$, $U_x=n\bar{x}$ and $U_y=m\bar{y}$.

1.2 $B<A<A+U_x<B+U_y$:

Then from (5″), (6) and (7) we have

$$\hat{R}=1-(r-1)U_x^{-1}\left(1+\frac{B}{U_y}\right)^{v-1}\left(1+\frac{A}{U_x}\right)^{r-2}\sum_{j=0}^{v-1}(-1)^j\frac{\binom{v-1}{j}}{(B+U_y)^j}$$

$$\times\int_A^{A+U_x}\left(1-\frac{\xi_x}{A+U_x}\right)^{r-2}\xi^j{}_x\,d\xi_x. \tag{10}$$

$$=1-(r-1)\,U_x^{-1}\left(1+\frac{B}{U_y}\right)^{v-1}\left(1+\frac{A}{U_x}\right)^{r-2}$$

$$\times\sum_{i=0}^{r-2}\sum_{j=0}^{v-1}(-1)^{i+j}\frac{\binom{r-2}{i}\binom{v-1}{j}}{(B+U_y)^j(A+U_x)^i}\cdot\frac{\{(A+U_x)^{i+j+1}-A^{i+j+1}\}}{(i+j+1)} \tag{11}$$

Tong's [6] result can be obtained putting in eq. (10) $A=B=0$, $r=n$, $v=m$, $U_x=n\bar{x}$ and $U_y=m\bar{y}$.

1.3 $A<B<A+U_x<B+U_y$:

Then from (5″), (6) and (7) we have

$$R=1-(r-1)U_x^{-1}\left(1+\frac{B}{U_y}\right)^{v-1}\left(1+\frac{A}{U_x}\right)^{r-2}$$

$$\times\sum_{i=0}^{r-2}\sum_{j=0}^{v-1}(-1)^{i+j}\frac{\binom{r-2}{i}\binom{v-1}{j}}{(A+U_x)^i(B+U_y)^j}\cdot\frac{\{(A+U_x)^{i+j+1}-B^{i+j+1}\}}{(i+j+1)} \tag{12}$$

1.4 $B<A<B+U_y<A+U_x$:

From (5″), (6) and (7) we have

$$\hat{R}=(v-1)U_y^{-1}\left(1+\frac{A}{U_x}\right)^{r-1}\left(1+\frac{B}{U_y}\right)^{v-2}$$

$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{v-2}(-1)^{i+j}\frac{\binom{r-1}{i}\binom{v-2}{j}}{(A+U_x)^i(B+U_y)^j}\cdot\frac{\{(B+U_y)^{i+j+1}-A^{i+j+1}\}}{(i+j+1)}. \tag{13}$$

2. *θ and μ are known, A and B are unknown* :

Here, $$\hat{R}=\int_0^\infty g(\xi_y \mid \hat{B}, \mu)\Big[\int_{\xi_y}^\infty f(\xi_x \mid \hat{A}, \theta)d\xi_x\Big]d\xi_y. \tag{5*}$$

Since, for known θ and μ ; $x_{(1)}$ and $y_{(1)}$ are the respective complete and sufficient estimators for A and B, therefore we find the conditional distributions $f(\xi_x \mid x_{(1)})$ and $g(\xi_y \mid y_{(1)})$ as :

$$f(\xi_x \mid x_{(1)})=\begin{cases} 1/n & , \text{ if } \xi_x=x_{(1)} \\ (1-\frac{1}{n})\frac{1}{\theta}\, exp\{-(\xi_x-x_{(1)})/\theta\}, & x_{(1)}<\xi_x\leqslant\infty \\ 0 & , \text{ otherwise} \end{cases} \tag{14}$$

$$g(\xi_y \mid y_{(1)})=\begin{cases} 1/m & , \text{ if } \xi_y=y_{(1)} \\ (1-\frac{1}{m})\frac{1}{\mu}\, exp\{-(\xi_y-y_{(1)})/\mu\}, & y_{(1)}<\xi_y\leqslant\infty \\ 0 & , \text{ otherwise.} \end{cases} \tag{15}$$

To obtain the MVUE, $\hat{R}_s$ the limits of ξ_x and ξ_y are taken as : $x_{(1)}<\xi_x\leqslant\infty$ and $y_{(1)}<\xi_y\leqslant\infty$. Here also two cases may be considered viz., (*i*) $y_{(1)}>x_{(1)}$ (*ii*) $x_{(1)}>y_{(1)}$

2.1 $y_{(1)}>x_{(1)}$:

Then from (5*), (14) and (15) we have

$$\hat{R}=\Big(1-\frac{1}{m}\Big)\Big(1-\frac{1}{n}\Big)\frac{\theta}{\theta+\mu}\, exp\,\{-(y_{(1)}-x_{(1)})/\theta\}. \tag{16}$$

2 2 $x_{(1)}>y_{(1)}$:

Then from (5*), (14) and (15) we have

$$\hat{R}=1-\Big(1-\frac{1}{m}\Big)\Big(1-\frac{1}{n}\Big)\frac{\mu}{\mu+\theta}\, exp\,\{-(x_{(1)}-y_{(1)})/\mu\}. \tag{17}$$

3. *(A, θ) and (B, μ) are unknown* :

The conditional distributions of $f(\xi_x \mid x_{(1)}, z_x)$ and $g(\xi_y \mid y_{(1)}, z_y)$ are :

$$f(\xi_x \mid x_{(1)}, z_x)=\begin{cases} 1/n & \text{if } \xi_r=x_{(1)} \\ \Big(1-\frac{1}{n}\Big)(r-2)\Big(1-\frac{\xi_x-x_{(1)}}{z_x}\Big)^{r-3}\frac{1}{z_x}, & x_{(1)}<\xi_x<x_{(1)}+z_x \\ 0 & , \text{ otherwise} \end{cases} \tag{18}$$

$$g(\xi_y \mid y_{(1)}, z_y) = \begin{cases} 1/m & \text{if } \xi_y = y_{(1)} \\ \left(1-\frac{1}{m}\right)(v-2)\left(1-\frac{\xi_y - y_{(1)}}{z_y}\right)^{v-3} \frac{1}{z_y}, & y_{(1)} < \xi_y < y_{(1)} + z_y. \\ 0 & \text{, otherwise.} \end{cases} \tag{19}$$

$\hat{R}$ is derived only for the cases when $x_{(1)} < \xi_x < x_{(1)} + z_x$ and $y_{(1)} < \xi_y < y_{(1)} + z_y$.

Here also four cases may arise viz.,

(i) $x_{(1)} < y_{(1)} < y_{(1)} + z_y < x_{(1)} + z_x$

(ii) $y_{(1)} < x_{(1)} < x_{(1)} + z_x < y_{(1)} + z_y$

(iii) $x_{(1)} < y_{(1)} < x_{(1)} + z_x < y_{(1)} + z_y$

(iv) $y_{(1)} < x_{(1)} < y_{(1)} + z_y < x_{(1)} + z_x$

Here we shall consider the above cases in the following subsections.

3.1. $x_{(1)} < y_{(1)} < y_{(1)} + z_y < x_{(1)} + z_x$:

Then from (5), (18) and (19) we have

$$\hat{R} = \left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\frac{v-2}{z_y}\left(1+\frac{x_{(1)}}{z_x}\right)^{r-2}\left(1+\frac{y_{(1)}}{z_y}\right)^{v-3}$$

$$\times \sum_{i=0}^{r-2}\sum_{j=0}^{v-3}(-1)^{i+j}\frac{\binom{r-2}{i}\binom{v-3}{j}}{(x_{(1)}+z_x)^i (y_{(1)}+z_y)^j}\cdot\frac{\{(y_{(1)}+z_y)^{i+j+1} - y_{(1)}^{i+j+1}\}}{(i+j+1)}. \tag{20}$$

3.2 $y_{(1)} < x_{(1)} < x_{(1)} + z_x < y_{(1)} + z_y$:

Then from (5′), (18) and (19) we have

$$\hat{R} = 1 - \left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\frac{r-2}{z_x}\left(1+\frac{x_{(1)}}{z_x}\right)^{r-3}\left(1+\frac{y_{(1)}}{z_y}\right)^{v-2}$$

$$\times \sum_{i=0}^{r-3}\sum_{j=0}^{v-2}(-1)^{i+j}\frac{\binom{r-3}{i}\binom{v-2}{j}}{(x_{(1)}+z_x)^i (y_{(1)}+z_y)^j}\cdot\frac{\{(x_{(1)}+z_x)^{i+j+1} - x_{(1)}^{i+j+1}\}}{(i+j+1)}. \tag{21}$$

3.3 $x_{(1)}<y_{(1)}<x_{(1)}+z_x<y_{(1)}+z_y$:

Then from (5′), (18) and (19) we have

$$\hat{R}=1-\left(1-\frac{1}{n}\right)\left(1-\frac{1}{m}\right)\frac{r-2}{z_x}\left(1+\frac{x_{(1)}}{z_x}\right)^{r-3}\left(1+\frac{y_{(1)}}{z_y}\right)^{v-2}\times$$

$$\times\sum_{i=0}^{r-3}\sum_{j=0}^{v-2}(-1)^{i+j}\frac{\binom{r-3}{i}\binom{v-2}{j}}{(x_{(1)}+z_x)^i\,(y_{(1)}+z_y)^j}\cdot\frac{\{(x_{(1)}+z_x)^{i+j+1}-y_{(1)}{}^{i+j+1}\}}{(i+j+1)}. \tag{22}$$

3.4 $y_{(1)}<x_{(1)}<y_{(1)}+z_y<x_{(1)}+z_x$:

Then from (5), (18) and (19) we get

$$\hat{R}=\left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\frac{v-2}{z_y}\left(1+\frac{x_{(1)}}{z_x}\right)^{r-2}\left(1+\frac{y_{(1)}}{z_y}\right)^{v-3}$$

$$\times\sum_{i=0}^{r-2}\sum_{j=0}^{v-3}(-1)^{i+j}\frac{\binom{r-2}{i}\binom{v-3}{j}}{(x_{(1)}+z_x)^i(y_{(1)}+z_y)^j}\cdot\frac{\{(y_{(1)}+z_y)^{i+j+1}-x_{(1)}{}^{i+j+1}\}}{(i+j+1)}. \tag{23}$$

4. *Maximum likelyhood estimator of R* :

The ML estimator of R can be obtained by substituting the ML estimates of A, B, θ and μ in (4). It is known (9) that if $\hat{\theta}$ is a ML estimator for θ then $U(\hat{\theta})$ is a ML estimator for $U(\theta)$. Here we consider the following three cases :

4.1 *A and B are known but θ and μ are unknown* :

The ML estimator of θ and μ are

$$\left.\begin{aligned}\hat{\theta}&=U_x/r\\ \hat{\mu}&=U_y/v\end{aligned}\right\} \tag{24}$$

4.2 *θ and μ are known but A and B are unknown :*

The ML estimator of A and B are

$$\left.\begin{aligned}\hat{A}&=x_{(1)}\\ \hat{B}&=y_{(1)}\end{aligned}\right\} \tag{25}$$

4.3 *(A, θ) and (B, μ) are unknown :*

The ML estimator of (A, θ) and (B, μ) are

$$\begin{aligned}\hat{A}&=x_{(1)}, \quad \hat{\theta}=\frac{z_x}{r}\\ \hat{B}&=y_{(1)}, \quad \hat{\mu}=\frac{z_y}{v}.\end{aligned} \tag{26}$$

Substituting the ML estimates (24) of θ and μ in (4) we canobtain the ML estimator of R for the case 4.1 as

$$\hat{R}=\frac{U_x v}{U_x v+V_y r}\ exp\{-(A-B)r/U_x\}. \tag{27}$$

Similarly the ML estimators of R for the case 4.2 and 4.3 are respectively given by

$$\hat{R}=\frac{\theta}{\theta+\mu}\ exp\{-(x_{(1)}-y_{(1)})/\theta\}. \tag{28}$$

$$\hat{R}=\frac{z_x v}{z_x v+z_y r}\ exp\{-r(x_{(1)}-y_{(1)})/z_x\}. \tag{29}$$

Illustrative Examples :

Let the available ordered samples from X and Y be (2, 6, 10, 14, 18) and (3, 4, 5, 6) respectively. For $n=6$ and $m=5$, we have $x_{(1)}=2$, $y_{(1)}=3$, $z_x=56$ and $z_y=9$. From (20) the MVU estimate of R is 0.537713 and from (29) the ML estimate of R is 0.761586.

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