Mathematical Forum, Vol. V, No. 2, September 1982, pp. 1-7

ON A DEGENERATE PARAMETRIC INTERACTION OF WAVES IN THERMOPIEZOELECTRIC MEDIA

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(Received on 15.7.1979)

ABSTRACT

The present paper is an attempt to investigate parametric amplification of waves in thermopielectric media for degenerate case. The theoretical analysis, as undertaken in this problem, makes use of the basic equation of mechanical, electrical and thermal continua, supplemented by appropriate constitutive relations. The conditions for amplification have been delineated.

1. *Introduction*

Studies in acoustics are replete with problems of propagation and, in particular ; those on amplification or attenuation of waves in a variety of media. Of all these media, piezoelectric media have achieved special attraction on account of a strong coupling between sound waves and electromagnetic waves in these media. Recent theoretical studies by Hutson and White [1], White [2], Musha [3] are few important evidences of such studies. More recently the studies on wave propagation have been extended further by reckoning thermal considerations affecting the constitutive character of piezoelectric materials, vide, Mindlin [4], Gupta and Sinha [5], [6]. These media are called thermopiezoelectric. All these studies, mentioned above, have been undertaken under small-signal assumption by linearizing nonlinear equations, whenever .arise. But if the excitations are not sufficiently weak, nonlinear effects may be appreciable and it would then be worthwhile to investigate parametric interactions and harmonic generation of waves in such cases. The present paper is an attempt in this direction and seeks to investigate collinear parametric interaction of waves in thermopiezoelectric media for degenerate case, i.e., for the case in which frequency of high frequency pump wave is twice the frequency of weak amplified wave. The present investigation shows that though the weak signal wave gets attenuated as it starts propagating, it is amplified after it traverses a certain distance if the pump wave is sufficiently intensive and the amplification co-efficient will be uniform for large distances.

2. *Statement of the problem and basic equations*

Let us consider a weak signal wave and a strong pump wave propagating through a thermopiezoelectric body. Our problem is to investigate application of signal wave by pump wave.

The basic equations of the present problem for a thermopiezoelectric medium are, vide, Mindlin [4] ; taking into account nonlinearity terms in the equation representing the thermal. These are

(i) the equation of mechanical motion, given by

$$\frac{\partial T}{\partial x}=\rho\,\frac{\partial^2(\delta x)}{\partial t^2}, \tag{2.1}$$

where T is the stress (mechanical), ρ the material density and δx the displacement (mechanical),

(ii) the equation of electric field, given by

$$\frac{\partial D}{\partial x}=0 \tag{2.2}$$

where D is the electric displacement.

(iii) the Fourier law of heat conduction, given by

$$\frac{k\partial^2\theta}{\partial x^2}=\Theta\frac{\partial\sigma}{\partial t} \tag{2.3}$$

where σ is the entropy density, k the heat conduction coefficient and $\Theta=\Theta_0+\theta$, Θ_0, θ being the reference and perturbed temperatures respectively.

(iv) the constitutive relation of the material, given by

$$T=C_1S-e_{pz}E-\lambda\theta, \tag{2.4}$$

$$D=\epsilon E+e_{pz}S+p\theta, \tag{2.5}$$

and

$$\sigma=\lambda S+pE+a\theta \tag{2.6}$$

where C_1 is the elastic constant, ϵ the dielectric permitivity, a the thermal constant, e_{pz} the piezoelectric constant, λ the thermoelastic constant and p the thermoelectric constant.

3. *Solution*

Let us now obtain a single differential equation governing the temperature θ from the above equation. To get this we first differentiate the equations (2.4) and (2.5) with respect to x and the equation (2.6) with respect to t and use the equations (2.1), (2.2) and (2.3). We then obtain

$$\rho\frac{\partial^2(\delta x)}{\partial t^2}=c_1\frac{\partial^2(\delta x)}{\partial x^2}-e_{pz}\frac{\partial E}{\partial x}-\lambda\frac{\partial\theta}{\partial x}, \tag{3.1}$$

$$\epsilon\frac{\partial E}{\partial x}+e_{pz}\frac{\partial^2(\delta x)}{\partial x^2}+p\frac{\partial\theta}{\partial x}=0, \tag{3.2}$$

and

$$k\frac{\partial^2\theta}{\partial x^2}=\Theta\left\{\lambda\frac{\partial^2(\delta x)}{\partial x\partial t}+p\frac{\partial E}{\partial t}+a\frac{\partial\theta}{\partial t}\right\}. \tag{3.3}$$

From the above three equations δx and E are eliminated to obtain the single differential equation in θ as

$$\frac{k\epsilon}{{\Theta_0}^2}(\Theta_0-\theta)\left\{\rho\epsilon\frac{\partial^3}{\partial t^2}-(c\epsilon+{e_{pz}}^2)\frac{\partial^2}{\partial x^2}\right\}\frac{\partial^3\theta}{\partial x^3}-\frac{k\epsilon}{{\Theta_0}^2}\left\{\epsilon\rho\frac{\partial^2\theta}{\partial t^2}\right.$$
$$\left.-(c\epsilon+{e_{pz}}^2)\frac{\partial^2\theta}{\partial x^2}\right\}\frac{\partial^3\theta}{\partial x^3}-\frac{k\epsilon}{{\Theta_0}^2}\left\{\rho\epsilon\frac{\partial^2}{\partial t^2}-(c\epsilon+{e_{pz}}^2)\frac{\partial^2}{\partial x^2}\right\}\left\{\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\right\}$$
$$-(a\epsilon-p^2)\left\{\epsilon\rho\frac{\partial^2}{\partial t^2}-(c\epsilon+{e_{pz}}^2)\frac{\partial^2}{\partial x^2}\right\}\frac{\partial^2\theta}{\partial x\partial t}+(\lambda\epsilon-pe_{pz})\frac{\partial^4\theta}{\partial x^3\partial t}=0. \tag{3.4}$$

First we consider the interaction of three waves whose (angular frequencies satisfy

$$\omega_3=\omega_1+\omega_2, \tag{3.5}$$

where ω_3 is the frequency of pump wave, ω_1 that of signal wave and ω_2 that of the wave generated in the medium.

As in Shiren [7], we now treat the interaction of waves by taking the temperature θ in the form

$$\theta=B_1e^{i(\omega_1t-k_1x)}+B_2e^{i(\omega_2t-k_2x)}$$
$$+B_3e^{i(\omega_3t-k_3x)}+c.\ c. \tag{3.6}$$

Let us now seek solution of the equation (3.4) by taking the variables and B's to be so slowly varying that we can retain only terms of the form $\frac{\partial B_i}{\partial x}$, B_iB_f, $B_i{B^*}_f$ etc. We then insert (3.6) in the equation (3.4) and equate terms of equal frequencies, giving rise so amplitude equatinos for coupled waves in the form, given by

$$M_1\frac{dB_1}{dx}+N_1B_1=P_1{B^*}_2B_3, \tag{3.7}$$

$$\bar{M}_2 \frac{dB_2}{dx} + N_2 B_2 = P_3 B^*_1 B_3, \tag{3.8}$$

and

$$M_3 \frac{dB_3}{dx} + N_3 B_3 = P_3 B_1 B_2, \tag{3.9}$$

where we have assumed that phase-matching condition is satisfied, i.e., $k_3 = k_1 + k_2$. In equations (3.7), (3.8) and (3.9) M's, N's and P's are quantities involving material parameters, frequencies and wave numbers. These are given by

$$M_1 = -\Big[\frac{k\epsilon}{\Theta_0}(c\epsilon + e_{pz}^2).\ 5k_1^4 + (a\epsilon - p^2).\ 3i\omega_1 k_1^2 + 3i\omega_1 k_1^2 (\lambda\epsilon - pe_{pz})^2\Big],$$

$$M_2 = -\Big[\frac{k\epsilon}{\Theta_0}(c\epsilon + e_{pz}^2).\ 5k_2^4 + (a\epsilon - p^2).\ 3i\omega_2 k_2^2 + 3i\omega_2 k_2^2 (\lambda\epsilon - pe_{pz})^2\Big]$$

$$M_3 = -\Big[\frac{k\epsilon}{\Theta_0}(c\epsilon + e_{pz}^2).\ 5k_3^4 + (\lambda\epsilon - pe_{pz})^2.\ 3i\omega_3 k_3^2 + (a\epsilon - p^2).\ 3i\omega_3 k_3^2\Big],$$

$$N_1 = -[(a\epsilon - p^2)\omega_1 k_1^3 + (\lambda\epsilon - pe_{pz})^2 \omega_1 k_1^3],$$

$$N_2 = -[(a\epsilon - p^2)\omega_2 k_2^3 + (\lambda\epsilon - pe_{pz})^2 \omega_2 k_2^3],$$

$$N_3 = -[(a\epsilon - p^2)\omega_3 k_3^3 + (\lambda\epsilon - pe_{pz})^2 \omega_3 k_3^3],$$

$$P_1 = \frac{k\epsilon e_{pz}^2}{\Theta_0^2}[i(k_3^5 - k_2^5) + ik_2^2 k_3^2 (k_3 - k_2) + (k_3^4 k_2^2 + k_2^4 k_3^2) - ik_3 k_2 (k_3^3 - k_2^3)],$$

$$P_2 = \frac{k\epsilon e_{pz}^2}{\Theta_0^2}[i(k_3^5 - k_1^5) + ik_1^2 k_3^2 (k_3 - k_1) + (k_3^4 k_1^2 + k_1^4 k_3^2) - ik_3 k_1 (k_3^3 - k_1^3)],$$

$$P_3 = \frac{k\epsilon e_{pz}^2}{\Theta_0^2}[i(k_1^5 + k_2^5) + ik_1^2 k_2^2 (k_1 + k_2) + k_1^2 k_2^2 (k_1^2 + k_2^2) + ik_1 k_2 (k_1^3 + k_2^3)].$$

Since M's and P's involve complex quantities, it is difficult to have analytical solution of the above equations. But we can get it if we take the parametric interaction to be of degenerate type, i.e., if we take

$\omega_1=\omega_2=\omega_{3/2}$. Then the equations (3.7) and (3.8) will be identical and setting $B_3=B_p$=constant, we get

$$M\frac{dB_1}{dx}+NB_1=PB_1{}^*B_p \tag{3.10}$$

where $M=M_1=M_2$, $N=N_1=N_2$ and $P=P_1=P_2$.

As done by Rudenko and Soluyan [8], we first take $B_1=A_1 \exp(is_1)$ and write $M=M'+iM''$, $P=P'+iP''$.

We then obtain from (3.10)

$$(M'^2+M''^2)\frac{dA_1}{dx}+NM'A_1=A_1B_p[(M'P'+M''P'')\cos 2s_1 + (M''P'-M'P'')\sin 2s_1], \tag{3.11}$$

and $$(M'^2+M''^2)\frac{ds_1}{d_x}-NM''=B_p[M'P'+M''P'')\sin 2s_1 - (M''P'-M'P'')\cos 2s_1], \tag{3.12}$$

where $$M'=-\frac{k\epsilon}{\Theta_0}(c\epsilon+e_{pz}{}^2)\,.\,5k_1{}^4=-ve,$$

$$M''=-\frac{3k\epsilon}{\Theta_0}[(a\epsilon-p^2)+(\lambda\epsilon-pe_{pz})^2]\omega_1k_1{}^3=-ve.$$

$$P'=\frac{k\epsilon e_{pz}{}^2}{\Theta_0{}^2}\,.\,20k_1{}^6=+ve,$$

$$P''=\frac{k\epsilon e_{pz}{}^2}{\Theta_0{}^2}\,.\,21k_1{}^5=+ve.$$

Let us now assume that

$$|\,(M''P'-M'P'')B_p+NM'\,|\;=a,$$
$$|\,2B_p(M'P'+M''P'')\,|\;=b,$$

and $$|\,(M''P'-M'P'')B_p-NM''\,|\;=c.$$

In order to obtain definite solution we assume that the pump wave is sufficiently strong and consider two cases :

$$M''P'>M'P'' \text{ and } M''P'<M'P''.$$

Case 1. When $M''P'>M'P''$, we have from (3.12) after integration

$$\tan S_1=-\frac{b}{2a}-\frac{\sqrt{b^2+4ac}}{2a}\left(\frac{1-ce^{-\alpha x}}{1+ce^{-\alpha x}}\right), \tag{3.13}$$

where

$$\alpha=\frac{\sqrt{b^2+4ac}}{M'^2+M''^2}$$

and $c = \frac{\sqrt{b^2+4ac}+b}{\sqrt{b^2+4ac}-b}$ (which is obtained by using the condition $x=0$, $S_1=0$).

Now if we put $A_1 = C_1 \exp(gx)$ in the equation (3·11) we easily get

$$g = \frac{1}{M'^2+M''^2}\left[-NM' - B_p\left(\frac{b}{2}\cos 2S_1 + a \sin 2S_1\right)\right]. \tag{3.14}$$

From (3.14), we see that for a sufficiently strong pump wave, g will be $+ve$, i.e., the amplification is possible only when $\tan 2S_1 < -\frac{b}{2a}$. From the equation (3.13) we can see that this condition must be satisfied when $x \geqslant \frac{1}{\alpha} \log c$.

Case II. When $M''P' < M'P''$, we get from (3.12)

$$\tan S_1 = \frac{b}{2a} + \frac{\sqrt{b^2+4ac}}{2a}\left(\frac{e^{\alpha x}-c}{e^{\alpha x}+c}\right). \tag{3.15}$$

Working out as before we can easily obtain the propagation distance for amplification and this is same as obtained earlier, i.e., $x \geqslant \frac{1}{\alpha} \log c$.

4. *Concluding Remarks*

From the results obtained in the earlier section, we can conclude that though the input signal wave is attenuated as it starts propagating, it is amplified after traversing certain distance if the pump wave is sufficiently strong and the amplification co-efficient will be uniform for large distance. Since the waves are collinear and are of the same mode type, the generation of the $\omega_4 (=\omega_3+\omega_1)$ acoustic wave will be phase matched for dispersionless acoustic waves and will be a stronger generation process than that of parametric interaction, vide, Nelson [9]. To prevent this phase matched generation at ω_4, a dispersion at frequency ω_4 is to be introduced, vide, Shiren [7]. Moreover growth of amplitude of weak wave will lead to the result that a reaction, i.e., inverse effect of signal wave on the pump will begin to play and we cannot regard B_p as constant after the beginning part of parametric amplification.

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