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APPROXIMATION OF FUNCTIONS ON [0, ∞] BY A NEW SEQUENCE OF MODIFIED SZÁSZ OPERATORS

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ABSTRACT

Here we define a new sequence of linear positive operators by modifying Szász operators with the weight function of Lupas operators on $L_1[0, \infty)$, the space of integrable functions defined on $[0, \infty)$ as

$$(G_{n,t}f)(x)=(n-1)\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty} P_{n,k}(y)f(x+y)\,dy,$$

where $p_{n,k}(v)=e^{-nv}\dfrac{(nv)^k}{k!}$ and $p_{n,k}(v)=\binom{n+k-1}{k}v^k(1+v)^{-n-k}$.

The present paper is a study of some results on approximation by these operators. In brief, we quote a few results as follows :

Let $d^rf(x+t)/dt^r$ be continuous on $[0, a]$, $d^r(G_{n,t}f)(x)/dt^r$ converges uniformly to $f^{(r)}(x+t)$ on $[0, a]$ and if f is continuous on $[0, \infty)$ then

$$\|(G_{n,t}f)(x)-f(x+t)\|_{C[0,a]}\leqslant\left\{1+\frac{2n(1+(n+3)a+3(n+1)a^2)}{(n-2)(n-3)}\right\}$$

$\omega(f; n^{-1/2})$, where $\omega(f;.)$ is the modulus of continuity of f.

Further, we prove that if $f\in L_1[0, \infty)$, $G_{n,t}f$ converges almost everywhere to f.

Introduction

Singh [1982] and Lupas [Papanicolan (1975)] had defined families of linear positlve operators mapping $C[0, \infty)$ into $C[0, \infty)$, the class of all bounded and continuous functions respectively as

$$(S_{n,t}f)(x)=\sum_{k=0}^{\infty} e^{-nt}\frac{(nt)^k}{k!}f\left(x+\frac{k}{n}\right) \tag{1.1}$$

and

$$(L_{n,t}f)(x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}t^k(1+t)^{-n-k}f\left(x+\frac{k}{n}\right), \tag{1.2}$$

where t ; $x\in[0, \infty)$ and x is fixed.

Motivated by the recent work of Derrienic [1981] on modified Bernotein Polynomials introduced by Durrmeyer [1967], we define a new family of positive linear operators, by modifying Sza'sz operators with the weight function of Lupas operators, for functions integrable on $[0, \infty)$ as

$$(G_{n,t}\, f)(x)=(n-1) \sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} P_{n,k}(y) f(x+y)\, dy, \quad (1.3)$$

where $p_{n,k}(v)=e^{-nv}\dfrac{(nv)^k}{k!}$ and $P_{n,k}(v)=\binom{n+k-1}{k} v^k(1+v)^{-n-k}$.

In the present paper, we study some results analogous to those in [2] on approximation by these operators.

Throughout our discussion, the superscript (r) and $\|\cdot\|$ stand for the rth derivative of the function involved and sup norm on $[0, \infty$ respectively.

2. First, we prove certain lemmas which will be used in the sequel.

For $m \in N^0$ (set of nonnegative integers), the mth order moment of the operators in (1.1) is defined by

$$U_{n,m}(t)=\sum_{k=0}^{\infty} p_{n,k}(t)\left(\frac{k}{n}-t\right)^m.$$

LEMMA 1. There holds the recurrence relation

$$nU_{n,m+1}(t)=t[U^{(1)}_{n,m}(t)+mU_{n,m-1}(t)].$$

Proof. It is easily observed that

$$t\,p^{(1)}_{n,k}(t)=(k-nt)\,p_{n,k}(t). \quad (2.1)$$

Hence the result.

Consequently,

(*a*) $U_{n,m}(t)$ is a polynomial in t of degree $\leqslant m$;

(*b*) $U_{n,m}(t)=0\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $[x]$ stands for the integral part of x.

LEMMA 2. The operator $(G_{n,t}\, f)(x)$ is a linear positive operator which maps 1 into 1 and transforms a polynomial of degree m into a polynomial of degree m.

Proof. Clearly, the operator $(G_{n,t}\, f)(x)$ is linear, positive and maps 1 into 1.

To pove the last assertion, without any loss of generality we can consider the polynomial as $f(y)=(y-x)^m$. Since

$$\int_0^{\infty} P_{n,k}(y) y^m dy=\frac{(m+k)!\,(n-m-2)!}{k!\,(n-1)!},$$

we have

$$(G_{n,t}(y-x)^m)(x)=(n-1)\sum_{k=0}^{\infty} P_{n,k}(t)\frac{(m+k)!\,(n-m-2)!}{k!\,(n-1)!}$$

$$=\frac{(n-m-2)!}{(n-2)!}\sum_{k=0}^{\infty} p_{n,k}(t)\frac{(m+k)!}{k!}.$$

For all $t,\ y\in[0,\infty)$

$$\frac{\partial^m}{\partial t^m}(t^m e^{-n(y-t)})=\sum_{k=0}^{\infty} e^{-ny}\frac{(nt)^k}{k!}\frac{(m+k)!}{k!}. \tag{2.3}$$

Applying Leibnitz theorem in the left part of (2 3), we get

$$\sum_{i=0}^{m}\binom{m}{i}\frac{m!}{i!}t^i n^i e^{-n(y=t)}. \tag{2.4}$$

Replacing y by t in (2.3) and (2.4), it follows that

$$\sum_{k=0}^{\infty} p_{n,k}(t)\frac{(m+k)!}{k!}=\sum_{i=0}^{m}\binom{m}{i}\frac{m!}{i!}t^i n^i. \tag{2.5}$$

Now, combining (2.2) and (2.5), we obtain the required result.

Further, we observe that

$$(G_{n,t}(y-x-t))(x)=\frac{1}{n-2}\sum_{k=0}^{\infty} p_{n,k}(t)(k+1)-t=\frac{nt+1}{n-2}-t$$

$$=\frac{(1+2t)}{(n-2)}. \tag{2.6}$$

LEMMA 3. Let the mth order moment for the operators $(G_{n,t}f)(x)$ be defined by

$$T_{n,m}(t)=(n-1)\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty} P_{n,k}(y)(y-t)^m\,dy.$$

Then

$$(n-m-2)T_{n,m+1}(t)=(m+1)(2t+1)T_{n,m}(t)+tT^{(1)}_{n,m}(t)$$
$$+mt(2+t)T_{n,m-l}(t). \tag{2.7}$$

Proof. Using (2.1), we get

$$tT^{(1)}_{n,m}(t)=(n-1)\sum_{k=0}^{\infty}(k-nt)p_{n,k}(t)\int_0^{\infty} P_{n,k}(y)(y-t)^m dy-mtT_{n,m-1}(t)$$

$$=(n-1)\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty}(k-ny)P_{n,k}(y)(y-t)^m\,dy+nT_{n,m+1}(t)$$
$$-mtT_{n,m-1}(t)$$

$$=(n-1)\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty} y(1+y)P^{(1)}_{n,k}(y)(y-t)^m dy+nT_{n,m+1}(t)$$
$$-mt\,T_{n,m-1}(t)$$

$$=(n-1)\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty}\{(y-t)^2+(2t+1)(y-t)+t)(1+t)\}$$

$$P^{(1)}_{n,k}(y)(y-t)^m dy+nT_{n,m+1}(t)-mt\,T_{n,m-1}(t)$$

$$=-(m+2)T_{n,m+1}(t)-(2t+1)(m+1)T_{n,m}(t)$$

$$-mt(1+t)T_{n,m-1}(t)+nT_{n,m+1}(t)-mt\,T_{n,m-2}(t).$$

Hence the lemma.

From (2.7) we conclude the following :

(a) $T_{n,m}(t)$ is a polynomial in t of degree m ;

(b) $T_{n,m}(t)=0\left(n^{-[\frac{m+1}{2}]}\right)$; and

(c) $T_{n,2}(t)=\dfrac{6(n+1)t^2+2(n+3)t+2}{(n-2)(n-3)}$, (in view of (2.6).

LEMMA 4. There exists the polynomials $q_{i,j,r}(t)$ independent of n and k, such that

$$t^r\frac{d^r}{dt^r}\{e^{-nt}(nt)^k\}=\sum_{\substack{2i+j\leq r\\ i,\,j\geq 0}} n^i(k-nt)^j q_{i,j,r}(t)e^{-nt}(nt)^k.$$

The proof of this lemma proceedes exactly on the lines of that of a result by Lorentz [Derrienic (1981)]

3. THEOREM 1. Let $f\in C(0,\infty)$, then

$$\|(G_{n,t}f)(x)-f(x+t)\|\leqslant\left\{1+\frac{2n(1+(n+3)a+3(n+1)a^2)}{(n-2)(n-3)}\right\}\omega(f;\,n^{-1/2})$$

where $\omega(f;\,.)$ is the modulus of continuity of f.

This theorem can easily be proved by using Lemma 3(c) and following a result due to Mond (1976).

THEOREM 2. Let f be integrable on $(0,\infty)$ and $f^{(1)}\in C[0,a]$. Then

$$\|(G_{n,t}f)(x)-f(x+t)\|\leqslant T_n\{\|f^{(1)}\|+2\omega(f^{(1)};\,T_n)\},$$

where $T_n=\|G_{n,t}(y-x-t)^2(x)\|^{1/2}$ and $\omega(f^{(1)};\,.)$ is the modulus of continuity of $f^{(1)}$.

The proof of this theorem, being on the lines of that of Censor (1971), is omitted.

THEOREM 3. Let f be bounded and integrable on $[0,\infty)$ and $f^{(2)}$ exists at a point $x+t\in[0,\infty)$, then

$$\lim_{n\infty} n[(G_{n,t}f)(x)-f(x+t)]=(1+2t)f^{(1)}(x+t)+t(t+3)f^{(2)}(x+t).$$

Further, this limit holds uniformly if $f^{(2)}$ is continuous on $[0,a]$.

Proof. By Taylor's expansion

$$f(y)=f(x+t)+(y-x-t)f^{(1)}(x+t)+\frac{(y-x-t)^2}{2}f^{(2)}(x+t)$$
$$+\varepsilon(y-x-t)(y-x-t)^2,$$

where $\varepsilon(y-x-t)$ is a bounded and integrable function on $[-x-t,\infty)$ tending to zero as $y\to x+t$.

Using (2·6) and Lemma 3(*c*), we have

$$n[(G_{n,t}f)(x)-f(x+t)]$$
$$=n\left\{\frac{(1+2t)}{(n-2)}f^{(1)}(x+t)+\frac{6(n+1)t^2+2(n+3)t+2}{2(n-2)(n-3)}.f^{(2)}(x+t)\right\}$$
$$+E(n,t),$$

where

$$E(n,t)=n(n-1)\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n,k}(y)\,\varepsilon(y-t)(y-t)^2dy. \tag{3.1}$$

Therefore to prove the result it is sufficient to show that $E(n,t)\to 0$, as $n\to\infty$. Since $\varepsilon(y-t)\to 0$ as $y\to t$, for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(y-t)|<\varepsilon$ whenever $0<|y-t|<\delta$.

Let $\phi_\delta(y)$ be the characteristic function of $y\in(t-\delta,t+\delta)$.

Then

$$\left|E(n,t)\right|\leqslant n(n-1)\sum_{k=0}^{\infty}p_{n,k}(t)\left\{\int_0^{\infty}P_{n,k}(y)\,\left|\varepsilon(y-t)\right|\phi_\delta(y)(y-t)^2dy\right.$$
$$\left.+\int_0^{\infty}P_{n,k}(y)\left|\varepsilon(y-t)\right|(1-\phi_\delta(y))(y-t)^2dy\right\}$$
$$=I_1+I_2 \text{ (say)}.$$

Evidently, in view of Lemma 3(*b*)

$$I_1<\varepsilon n(n-1)\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n,k}(y)(y-t)^2dy$$
$$=\varepsilon.0(1)$$

and

$$I_2\leqslant M\,n(n-1)\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n,k}(y)\frac{(y-t)^{2q}}{\delta^{2q-2}}dy\ ;\ q\text{ (integer)}\geqslant 2$$
$$=0(n^{-(q-1)}),\ M=\sup_{u\in[-t,\infty)}|\varepsilon(u)|.$$

combining the estimates of I_1 and I_2, due to the arbitrariness of $\varepsilon>0$ it follows that $E(n,t)\to 0$ for sufficiently large n.

To prove the uniformity assertion, since $f^{(2)}$ is continuous on $[0,a]$, we have

$$|\epsilon(y-t)|\leqslant\omega(f^{(2)};|y-t|)\leqslant\left(1+\frac{|y-t|}{\delta}\right)\omega(f^{(2)};\delta)$$

for all $\delta>0$

using the above inequality in (3.1), we get

$$| E(n, t) | \leqslant n(n-1)\omega(f^{(2)}; \delta)[\sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} P_{n,k}(y)(y-t)^2 dy$$

$$+\frac{1}{\delta} \sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} P_{n,k}(y) \mid y-t|^3 dy]$$

$$= I_3 + I_4 \text{ (let).}$$

Let us estimate I_4 first.

Application of Schwartz's inequality for summation and integral and Lemma 3(b) leads to

$$I_4 \leqslant n(n-1)\frac{\omega(f^{(2)}; \delta}{\delta}\{(\sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} P_{n,k}(y)(y-t)^2 dy)^{1/2}.$$

$$(\sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} P_{n,k}(y)(y-t)^4 dy^{1/2}\}$$

$$= \omega(f^{(2)}; \delta)\, 0(n^{-1/2}.\delta^{-1}).$$

Further, use of Lemma 3(b) yields

$$I_3 = \omega(f^{(2)}; \delta).0(1).$$

Now, choosing $\delta = n^{-1/2}$ and noting that in view of Lemma 3(a)) the moments for the operators $G_{n,t}$ are polynomials in t, we obtain

$$| E(n, t) | \leqslant M_1 \omega(f^{(2)}; n^{-1/2}).$$

The proof is complete.

Theorem 4. Let f be a bounded and integrable function on $[0, \infty\}$ admitting a derivative of order r at $x+t \in [0, \infty)$. Then

$$\lim_{n\to\infty} (G_{n,t}^{(r)} f)(x) = f^{(r)}(x+t).$$

Proof. Application of Leibnitz theorem yields

$$(G_{n,t}^{(r)} f)(x) = (n-1) \sum_{k=0}^{\infty} \frac{d^r}{dt^r}\left\{e^{-nt}\frac{(nt)^k}{k!}\right\} \int_0^{\infty} P_{n,k}(y) f(x+y)\, dy$$

$$= (n-1) \sum_{i=0}^{r} \sum_{k=i}^{\infty} \binom{r}{i} \frac{(-1)^{r-i} n^r e^{-nt} (nt)^{k-i}}{(k-i)!} \int_0^{\infty} P_{n,k}(y) f(x+y)\, dy$$

$$= (n-1) \sum_{k=0}^{\infty} \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} n^r P_{n,k}(t) \int_0^{\infty} P_{n,k+i}(y) f(x+y) dy$$

$$= (n-1) \sum_{k=0}^{\infty} p_{n,k}(t) \int_0^{\infty} (-1)^r \Big(\sum_{i=0}^{r} \binom{r}{i}(-1)^i n^r P_{n,k+i}(y)\Big)$$

$$f(x+y)\, dy.$$

Since by Leibnitz theorem

$$P_{n-r,k+r}^{(r)}(y) = \frac{(n-1)!}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} P_{n,k+i}(y).$$

Therefore,

$$(G^{(r)}_{n,t}f)(x)=\frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty}(-1)^r P^{(r)}_{n-r,k+r}(y)f(x+y)\,dy.$$

Now, by Taylor's expansion of the function f,

$$f(y)=\sum_{i=0}^{r}\frac{(y-x-t)^i}{i!}f^{(i)}(x+t)+\varepsilon(y-x-t)(y-x-t)^r$$

$$=R(y)+\varepsilon(y-x-t)(y-x-t)^r \qquad \text{(say)},$$

where $\varepsilon(y-x-t)\to 0$ as $y\to x+t$ and is a bounded and integrable function on $(-x-t,\infty)$.

On integrating r times by parts, we obtain

$$(G^{(r)}_{n,t}R)x=\frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^{\infty} p_{n,k}(t)\int_0^{\infty} p_{n-r,k+r}(y)\,R^{(r)}(x+y)\,dy$$

clearly

$$R^{(r)}(x+y)=f^{(r)}(x+t).$$

Hence

$$(G^{(r)}_{n,t}R)(x)=\frac{n^r(n-r-2)!}{(n-2)!}f^{(r)}(x+t)\to f^{(r)}(x+t)$$

as $n\to\infty$.

Now, it remains to show that

$$(G^{(r)}_{n,t}(f-R))(x)\to 0 \text{ as } n\to\infty,$$

where $(f-R)(y)=\varepsilon(y-x-t)(y-x-t)^r$.

Applying Lemma 4, we get

$$t^r(G^{(r)}_{n,t}(f-R))(x)=(n-1)\sum_{\substack{2i+j\leq r\\ i,\ j\geq 0}} n^i q_{i,j,r}(t)\sum_{k=0}^{\infty}(k-nt)^j p_{n,k}(t).$$

$$\int_0^{\infty} P_{n,k}(y)\,\varepsilon(y-t)(y-t)^r\,dy.$$

By using Schwartz inequality for summation and integral, we have

$$I\equiv|t^r(G^{(r)}_{n,t}(f-R))(x)|\leq M_2(n-1)\sum_{\substack{2i+j\leq r\\ i,\ j\geq 0}} n^i\Big(\sum_{k=0}^{\infty}(k-nt)^{2j}p_{n,k}(t)\Big)^{1/2}.$$

$$\Big(\sum_{k=0}^{\infty} p_{n,k}(t)\Big(\int_0^{\infty} P_{n,k}(y)\,\varepsilon(y-t)\,(y-t)^r dy\Big)^2\Big)^{1/2}.$$

Since $\varepsilon(y-t)\to 0$ as $y\to t$, hence for a given $\epsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(y-t)|<\varepsilon$ whenever $|y-t|<\delta$. For $|y-t|\geq\delta$, the boundedness of $\varepsilon(y-t)$ implies

$$|\varepsilon(y-t)|\leq\frac{M_3}{\delta}|y-t|,\quad M_3=\sup_{u\in[-t,\infty)}|\varepsilon(u)|. \tag{3.2}$$

Thus, finally we obtain

$$\{\varepsilon(y-t)^2 < \varepsilon^2 + \frac{M_3^2}{\delta^2}(y-t)^2 \text{ for all } y \in [0, \infty).$$

Therefore, using Schwartz inequality and (3.2), we have

$$\left(\int_0^\infty P_{n,k}(y)\,\varepsilon(y-t)\,(y-t)^r dy\right)^2$$

$$\leqslant \frac{1}{n-1}\int_0^\infty P_{n,k}(y)\{\varepsilon^2(y-t)^{2r} + \frac{M_3^2}{\delta^2}(y-t)^{2r+2}\}\,dy.$$

Now, using Lemma 3(b), it follows that

$$\sum_{k=0}^\infty p_{n,k}(t)\left(\int_0^\infty P_{n,k}(y)\,\epsilon(y-t)(y-t)^r dy\right)^2$$

$$\leqslant \frac{1}{n-1}\sum_{k=0}^\infty p_{n,k}(t)\int_0^\infty P_{n,k}(y)\{\varepsilon^2(y-t)^{2r} + \frac{M_3^2}{\delta^2}(y-t)^{2r+2}\}\,dy$$

$$= \varepsilon^2\, 0(n^{-(r+2)}).$$

Lastly, using Lemma 1(b)

$$I \leqslant M_2(n-1) \sum_{\substack{2i+j\leqslant r \\ i,\ j\geqslant 0}} 0\left(n^{i+(\frac{j}{2})}\right).\ \varepsilon\ 0(n^{-(r+2/2)})$$

$$\leqslant M_4\varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$, $I \to 0$ for sufficiently large n. This completes the proof.

Theorem 5. If $f^{(r)}(x+t)$ is continuous on $[0, a]$, $(G_{n,t}^{(r)} f)(x)$ converges uniformly to $f^{(r)}(x+t)$ and

$$\left\| \frac{(n-2)!}{n^r(n-r-2)!}(G_{n,t}^{(r)} f)(x) - f^{(r)}(x+t) \right\| \leqslant k_r\,\omega(f^{(r)};\ n^{-1/2}), \quad (3.3)$$

where k_r is a constant independent of f and n and $\omega(f^{(r)};\ .)$ is the modulus of continuity of rth derivative of f.

Proof. We have

$$(G_{n,t}^{(r)} f)(x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^\infty P_{n,k}(t)\int_0^\infty (-1)^r P_{n-r,k+r}^{(r)}(y)\, f(x+y)\,dy$$

$$= \frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^\infty p_{n,k}(t)\int_0^\infty P_{n-r,k+r}(y)^{(r)}(x+y)\,dy.$$

Hence, by using Schwartz inequality for summation and integral and Lemma 3(b), we get

$$\left| \frac{(n-2)!}{n^r(n-r-2)!} G_{n,t}^{(r)} f)(x) - f^{(r)}(x+t) \right|$$

$$\leqslant(n-r-1)\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n-r,k+r}(y)\,|f^{(r)}(x+y)-f^{(r)}(x+t|\,dy$$

$$\leqslant\omega(f^{(r)};\delta)(n-r-1)\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n-r,k+r}(y)\,(1+\frac{|y-t|}{\delta}\,dy$$

$$\leqslant\omega(f^{(r)};\delta)\left[1+\frac{\sqrt{n-r-1}}{\delta}(\sum_{k=0}^{\infty}p_{n,k}(t)\int_0^{\infty}P_{n-r,k+r}(y)(y-t)^2\,dy)^{1/2}\right]$$

$$\leqslant\omega(f^{(r)};\delta)\left[1+\frac{1}{\delta}0(n^{-1/2})\right].$$

Choosing $\delta=n^{-1}/^2$, (3.3) follows.

The convergence of $(G_{n,t}^{(r)}f)(x)$ to $f^{(r)}\{x+t)$ is uniform, since $f^{(r)}(x+t)$ is bounded on $[0,a]$ and

$$\frac{(n-2)!}{n^r(n-r-2)!}\rightarrow 1 \text{ as } n\leftarrow\infty.$$

THEOREM 6. *Let f be an integrable function on $[0,\infty)$. Then, the sequence $G_{n,t}f$ converges to f almost everywhere.*

Proof. Let $F(x+y)=\int_0^y f(x+u)\,du$, then $F'(x+y)=f(x+y)$ almost everywhere in $(0,\infty)$. Let $a\Theta(0,\infty)$ where $F'(x+a)=f(x+a)$. Now in order to prove the theorem, it is sufficient to show that $(G_{n,a}f)(x)$ converges to $f(x+a)$.

For all k, $0\leqslant k<\infty$, the product $P'_{n,k}(y)F(x+y)$ is absolutely continuous and because $F'(x+y)=f(x+y)$ almost everywhere we have for all $t\Theta[0,\infty]$:

$$P_{n,k}(t)F(x+t)=\int_0^t P'_{n,k}(y)F(x+y)dy+\int_0^t P_{n,k}(y)f(x+y)\,dy. \tag{3·4}$$

Hence

$$G_{n,a}f)(x)=-(n-1)\sum_{k=0}^{\infty}(p_{n,k}(a)\int_0^{\infty}P'_{n,k}(y)F(x+y)\,dy. \tag{3.5}$$

The function F being differentiable at $x+a$, we have

$$F(x+y)=F(x+a)+(y-a)F'(x+a)+(y-a)\,\varepsilon\,(y-a),$$

where $\varepsilon(u)\rightarrow 0$ as $u\rightarrow 0$ and $\varepsilon(u)$ is bounded. We put

$$M=\sup{}_{[-a,\infty]}\,|\,\varepsilon(y-a)\,|\,.$$

Using (3.4) in (3.5), we get

$$(G_{n,a}f)(x)=(n-1)e^{-na}F(x+a)-(n-1)ae^{-na}F'(x+a)+F'(x+a)$$
$$-(n-1)\sum_{k=0}^{\infty}p_{n,k}(a)\int_0^{\infty}P'_{n,k}(y)(y-a)\,\varepsilon\,(y-a)\,dy.$$

Now, to show that $(G_{n,a}f)(x)$ converges to $f(x+a)$, it is sufficient to prove that

$$R_n(a)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(a)\int_0^{\infty}P'_{n,k}(y)(y-a)\,\varepsilon\,(y-a)dy\rightarrow 0 \text{ as } n\rightarrow\infty.$$

On utilizing

$$P'_{n,k}(y)=n[P_{n+1,k-1}(y)-P_{n+1,k}(y)]$$

in $R_n(a)$, we have

$$R_n(a)=n(n-1)\sum_{k=0}^{\infty} p_{n,k}(a)\int_0^{\infty}[P_{n+1,k-1}(y)-P_{n+1,k}(y)](y-a)\,\varepsilon(y-a)\,dy$$

$$=n(n-1)\sum_{k=0}^{\infty}(p_{n,k+1}(a)-p_{n,k}(a))\int_0^{\infty}P_{n+1,k}(y)(y-a)\,\varepsilon(y-a)\,dy.$$

Further simplification leads us to

$$aR_n(a)=(n-1)\sum_{k=0}^{\infty} p_{n,k+1}(a)(na-(k+1))\int_0^{\infty}P_{n+1,k}(y)(y-a)\,\varepsilon(y-a)\,dy.$$

Using Schwartz's inequality for summation, we obtain

$$(aR_n(a))^2\leqslant(n-1)^2\sum_{k=0}^{\infty}P_{n,k+1}(a)(n\,a-(k+1))^2$$

$$x(\sum_{k=0}^{\infty}p_{n,k+1}(a)(\int_0^{\infty}P_{n+1,k}(y)(y-a)\,\epsilon(y-a)\,dy)^2).$$

Now, using Schwartz's inequality for integral

$$\sum_{k=0}^{\infty}P_{n,\,k+1}(a)\,(\int_0^{\infty}P_{n+1,\,k}(y)(y-a)\,\varepsilon(y-a)\,dy)^2$$

$$\leqslant\sum_{k=0}^{\infty}P_{n,\,k+1}(a)(\int_0^{\infty}P_{n+1,\,k}(y)dy)(\int_0^{\infty}P_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy)$$

$$=\frac{1}{n}\sum_{k=0}^{\infty}P_{n,\,k+1}(y)(\int_0^{\infty}P_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy)$$

$$=\frac{1}{n}\sum_{k=0}^{\infty}P_{n,\,k+1}(a)[\int_{|y-a|<\delta}p_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy$$

$$+\int_{|y-a|\geqslant\delta}P_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy]$$

$$=I_1+I_2,$$

where

$$I_1=\frac{1}{n}\sum_{k=0}^{\infty}p_{n,\,k+1}(a)\int_{|y-a|<\delta}P_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy$$

$$<\varepsilon^2\frac{1}{n}\sum_{k=0}^{\infty}p_{n,\,k+1}(a)\int_0^{\infty}P_{n+1,\,k}(y)(y-a)^2dy$$

$$=\varepsilon^2\ 0\left(\frac{1}{n^3}\right),$$

$$I_2=\frac{1}{n}\sum_{k=0}^{\infty}p_{n,\,k+1}(a)\int_{|y-a|\geqslant\delta}P_{n+1,\,k}(y)(y-a)^2(\varepsilon(y-a))^2dy$$

$$\leqslant\frac{M^2}{n\delta^2}\sum_{k=0}^{\infty}p_{n,\,k+1}(a)\int_0^{\infty}P_{n+1,\,k}(y)(y-a)^4dy$$

$$=\frac{1}{\delta^2}0\left(\frac{1}{n^4}\right).$$

Consequently,

$$\sum_{k=0}^{\infty} p_{n,\, k+1}(a)\Big(\int_0^{\infty} P_{n+1,\, k}(y)(y-a)\,(\varepsilon(y-a))\, dy\Big)^2$$

$$=\Big(\varepsilon^2+\frac{1}{n\delta^2}\Big)0\Big(\frac{1}{n^3}\Big)$$

$$=\varepsilon^2\, 0\Big(\frac{1}{n^3}\Big).$$

Also, by Lemma 1

$$\sum_{k=0}^{\infty} p_{n,\, k}(t)(k-nt)^2 = nt\,(1+t),$$

so $\frac{1}{n}\sum_{k=0}^{\infty} p_{n,k+1}(a)(na-(k+1))^2$ is bounded uniformly in n.

Hence, for sufficiently large n and $\varepsilon>0$, we have

$$aR_n(a) \leqslant K\varepsilon,$$

where K is a constant independent of n and ε.

This completes the proof.

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