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ON GENERALISED RECURRENT KÄEHLER MANIFOLDS

by

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ABSTRACT

Recurrent Käehler manifolds have been studied by Lal and Singh [2] and Singh [3, 4, 5]. In this paper we define a generalized recurrent Kaehler manifold and study some of its properties.

1. *Introduction*

Let K^n be a Käehlerian manifold of real dimension n $(=2m)$ with local coordinates x^i. We shall restrict our attention in the present paper to manifolds which are real representations of complex Käehlerian manifolds. (Indices run over the range from 1 to n). Then the positive definite Riemannian metric g and the complex structure J satisfy the relations

$$J^r_j J^i_r = -\delta^i_j, \; g_{rt} J^r_j J^t_i = g_{ji}, \tag{1.1}$$

$$J_{ji} = J^r_j g_{ri} = -J_{ij}, \; J^{ji} = g^{jr} J^i_r = -J^{ij},$$

$$J^i_{j,k} = 0, \; g_{ji,k} = 0,$$

where a comma denotes the operator of covariant differentiation with respect to the Riemannian connection.

Let $R^h{}_{kji}$ be the Riemannian curvature tensor. The Ricci tensor and the scalar curvature are respectively given by

$$R_{ji} = R^a_{aji} \text{ and } R = R_{ji} g^{ji}.$$

We define a tensor S_{ji} by

$$S_{ji} = J^r_j R_{ri}, \tag{1.2}$$

then we have

$$S_{ji} = -S_{ij} \tag{1.3}$$

and

$$R^r_{kji} J^h_r = R^h_{kjr} J^r_i, \; R_{kjir} J^r_h = R_{kjhr} J^r_i, \tag{1.4}$$

where $R_{kjih} = R^a_{kji} g_{ah}$.

K^n is called Käehlerian recurrent if its Riemannian curvature tensor R_{hijk} satisfies

$$R_{hijk,m}=K_m R_{hijk}, \tag{1.5}$$

for some non-zero vector K_m.

We shall consider the following generalization.

Definition 1.1 : A Kaehler manifold of dimension n shall be called a generalized recurrent Kaehler manifold, denoted by GK^n, if there exist two non-zero vectors K_m and L_m such that

$$R_{hijk,m}=K_m R_{hijk}+L_m G_{hijk}, \tag{1.6}$$

where

$$G_{hijk}=g_{ki}g_{jh}-g_{ji}g_{kh}+J_{ki}J_{jh}-J_{ji}J_{kh}+2J_{kj}J_{ih}. \tag{1.7}$$

The vectors K_m and L_m are associated vectors of recurrence. The tensors G_{hijk} satisfies all the algebraic identities satisfy by R_{hijk} In case $L_m=0$, the space reduces to a recurrent Kaehler manifold characterized by (1.5).

2. *Some Properties of Generalized Recurrent Kaehler Manifolds.*

Multiplying (1.6) by $g^{ij}\, g^{hk}$ we get

$$R_{,m}=K_m R-n(n+2)L_m. \tag{2.1}$$

Eliminating L_m between (1.6) and (2.1) we get

$$U_{hijk,m}=K_m U_{hijk}, \tag{2.2}$$

where

$$U_{hijk}=R_{hijk}+R/n(n+2)\, G_{hijk}. \tag{2.3}$$

Conversely, if (2.2) holds, we can define a vector $L_m=(K_m R-R_{,m})/n(n+2)$ such that (1.6) holds. Thus we have

Theorem 2.1 : A necessary and sufficient condition for a Kaehler manifold K^n to be a GK^n is that the tensor U_{hijk} given by (2.3) is recurrent.

It may be noted thar (1.5) implies (2.2). But the converse is not necessarily true. We thereby have

Corollary 2.1 Every recurrent K^n is a GK^n but the converse is not necessarily true.

Hasegawa [1] has called the tensor U_{hijk} given by (2.3) the H-concircular curvature tensor of K^n. We may therefore define a GK^n in an alternative form as follows : A Kaehler manifold whose H-concircular curvature tensor is recurrent is generalized recurrent Kaehler manifold.

From (2.1) we observe that $L_m=0$ if and only if $R_{,m}-K_m R=0$. Hence we shall assume that $R_{,m}-RK_m\neq 0$.

Again if $U_{hijk}=0$, then (2.2) is satisfied identically. But $U_{hijk}=0$ implies that the Kaëhler manifold is that of constant holomorphic sectional curvature. Thus a Kaëhler manifold of constant holomorphic sectional curvature is a trivial example of a generalized recurrent Kaëhler manifold.

Multiplying U_{hijk} by g^{hk} we get

$$U_{ij}=U_{hijk}\,g^{hk}=R_{ij}-(R/n)g_{ij}. \tag{2.4}$$

Hence we have

Theorem 2.2. In a GK^n the tensor U_{ij} is recurrent.

Proof: The proof follows from (2.2) and (2.4).

Theorem 2.3. A necessary condition for a Gk^n to be recurrent is given by

$$R=0 \tag{2.5}$$

or

$$R,_{m}=RK_{m}. \tag{2.6}$$

Proof. The result follows from (1.6), (2.1), (2.5) and (2.6).

Now we shall prove that the associated vectors of recurrence are unique.

If possible let K_m and L_m be the two vectors satisfying (1.6) then

$$R_{hijk},_{m}=K'_{m}R_{hijk}+L'_{m}G_{hijk}. \tag{2.7}$$

Subtracting (2.7) from (1.6) we obtain

$$K^{*}_{m}\,R_{hijk}+L^{*}_{m}\,G_{hijk}=0 \tag{2.8}$$

where

$$K^{*}_{m}=K_{m}-K'_{m},\ L^{*}_{m}=L_{m}-L'_{m}. \tag{2.9}$$

From (2.8) we observe that if $K^{*}_{m}=0$ then $L^{*}_{m}=0$ and conversely. Thus if K_m is unique then L_m is also unique and conversely.

If $K^{*}_{m}\neq 0$ we can choose a vector a^m satisfying $K^{*}_{m}\,a^m=1$ so that in view of (2.8) we obtain $R_{hijk}=KG_{hijk}$ with $K=-L^{*}_{m}\,a^m$, showing that the manifold is that of constant holomorphic sectional curvature. We thereby have the following.

Theorem 2.4. In a GK^n the associated vectors of recurrence are unique or the space is that of constant holomorphic sectional curvature.

We shall use the following Lemmas as given in [7,4].

Lemma I. (A. G. Walker). The curvature tensor of a Riemannian space satisfies the identity

$$(R_{hijk},_{lm}-R_{hijk},_{ml})+(R_{jklm},_{hi}-R_{jklm},_{ih})$$
$$+(R_{lmhi},_{jk}-R_{lmhi},_{kj})=0, \tag{2.10}$$

where

$$R_{hijk,lm}=R_{hijk,l,m}.$$

Lemma II. (A. G. Walker). If a_{ij} and b_i are numbers satisfying $a_{ij}=a_{ji}$ and $a_{ij}b_k+a_{jk}b_i+a_{ki}b_j=0$, then either all the a_{ij} are zero or all the b_i are zero.

It may be verified that U_{hijk} also satisfies the identity (2.10). Substituting U_{hijk} for R_{hijk} in (2 10) and using (2.2) we obtain

$$U_{hijh}K_{lm}-U_{jklm}K_{hi}+U_{lmhi}K_{jk}=0,$$

where $K_{lm}=K_{l,m}-K_{m,l}$. Using Walker's Lemma II we immediately deduce that $K_{lm}=0$, showing that K_m is irrotational.

Hence we have

Theorem 2.5 : In a GK^n the recurrence vector K_m is irrotational.

Theorem 2.6 : L_m is not a gradient vector except in the case when $R=$constant.

Proof: The result follows from (2.1) and Theorem 2.5.

Lemma III. If T be any recurrent tensor whose recurrence vector is gradient of some scalar function f, then $(e^{-f}T)$ is covariant constant.

Proof: The statement can be verified directly. Hence in view of Theorems 2.1, 2.5 and Lemma III we have the following.

Theorem 2.7 : In a GK^n if $K_m=f,m$ then $(e^{-f}U_{hijk})$ is covariant constant.

3. *GK^n satisfying $R=$constant*

If $R=$constant, then in view of the well known relation

$$R_{,i}=2R^a_{i,a}$$

and from (2.4) we obtain

$$K_aU^a_i=0, \tag{3.1}$$

where $\quad U^a_i-U_{ij}g^{ja}.$

Equation (3.1) immediately yields

$$K_aR^a_i=\frac{R}{n}K_i. \tag{3.2}$$

This can easily be given the form of a recursive formula.

Theorem 3.1 : In a GK^n with $R=$constant we have

$$K^{(p)}_a\ R^a_j=\left(\frac{R}{n}\right)^pK_j, \tag{3,3}$$

p being any positive integer.

Proof: Let us define

$$(p)R^i_j={}^{(p-1)}R^i_aR^a_j,\ p=2,3,\cdots\cdots$$
$${}^{(0)}R^i_i=\delta^i_j.$$

Multiplying (3.2) by R^i_j and using it again we obtain

$$^{(2)}R^a_j K_a = \left(\frac{R}{n}\right)^2 K_j .$$

The result follows by induction.

Theorem 3.2 : In a GK^n with R=constant we have

$$U_{hijk}K_l + U_{hikl}K_j + U_{hilj}K_k = 0, \tag{3.4}$$

$$K_a U^a_{ijk} = U_{ij}K_k - U_{ik}K_j \tag{3.5}$$

and

$$U_{hijk} = U_{hk}p_i p_j + U_{ij}p_h p_k - U_{ik}p_j p_h - U_{jh}p_i p_k, \tag{3.6}$$

where $\quad p_i = k_i/(K_m K^m)^{\frac{1}{2}}$.

Proof : (3.4) follows from (2.2) and Bianchi's identities on R_{hijk}. Multiplying (3.4) by g^{hl} and taking into consideration (2.4) and the symmetrics of U_{hijk} we obtain (3.5).

Again (3.6) follows from (3.4), (3.5) and (3.1) by straight forward computation.

Theorem 3.3 : In a GK^n with R=constant the vectors K_m and L_m are collinear.

Proof : This follows from (2.1).

4. *Some examples of generalized recurrent Kaëhler manifolds.*

In a Kaëhler manifold, the holomorphically projective curvature tensor and the Bochner curvature tensor are respectively given by

$$P_{hijk} = R_{hijk} + \frac{1}{n+2}[R_{hi}g_{ik} - R_{ij}g_{hk} + S_{hj}J_{ik} - S_{ij}J_{hk} + 2S_{hi}J_{jk} \tag{4.1}$$

and

$$\begin{aligned} B_{hijk} = R_{hijk} &+ \frac{1}{n+4}[R_{hj}g_{ik} - R_{ij}g_{hk} + g_{hj}R_{ik} - g_{ij}R_{hk} \\ &+ S_{hj}J_{ik} - S_{ij}J_{hk} + J_{hj}S_{ik} - J_{ij}S_{hk} + 2S_{hi}J_{jk} \\ &+ 2S_{hi}S_{jk}] - \frac{R}{(n+2)(n+4)}[g_{hj}g_{ik} - g_{ij}g_{hk} + J_{hj}J_{ik} \\ &- J_{ij}J_{hk} + 2J_{hi}J_{jk}]. \end{aligned} \tag{4.2}$$

It is well known that if a Kaëhler manifold is an Einstein one, then the H-projective curvature tensor and the Bochner curvature tensor reduces to the tensor U_{hijk} given by (2.3). Hence in view of Theorem 2.1 we have

Theorem 4.1 : An Einstein Käehler manifold with recurrent holomorphically projective curvature tensor is a GK^n.

Theorem 4.2 : An Einstein Käehler manifold with recurrent Bochner curvature tensor is a GK^n.

The following Theorem is known [1]

Theorem 4.3 : A necessary and sufficient condition for a Käehlerian manifold K^n to be H-projective recurrent is that K^n be H-concircular recurrent.

We thereby have

Theorem 4.4 : A necessary and sufficient condition for the manifold K^n to be GK^n is that K^n be H-projective recurrent.

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