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DERIVATION OF LOGISTIC LAW OF POPULATION GROWTH FROM MAXIMUM-ENTROPY PRINCIPLE

By

J. N. KAPUR

Indian Institute of Technology, Kanpur

ABSTRACT

In the present note, we combine the use of the calculus of variations in population dynamics by Volterra (1928) with the measures of entropy introduced by Shannon (1948) and Kapur (1972, 1983a, b) in order to deduce the exponential and logistic laws of population growth of Malthus (1798), Verhulst (1845) and Pearl and Reed (1920) from the maximum-entropy principle enunciated by Jaynes (1957).

1. *The Maximum-Entropy Principle*

According to the maximum entropy principle, the population N(t) should change with time as little as possible, subject to any constraints that may be imposed on it by the available information, unless the available information is sufficient to determine N(t) completely.

Thus if the only available information is that the time-average of the population is $\overline{N}$, then we maximize

$$-\int_a^b N(t)\ \ln N(t)\, dt \tag{1}$$

subject to

$$\int_a^b N(t)\, dt \Big/ \int_a^b dt = \overline{N}, \tag{2}$$

to get

$$N(t) = \overline{N}, \tag{3}$$

so that in this case, the population remains constant.

2. *Malthus Law of Population Growth*

Now let the constraint be

$$\int_a^b dt \int_0^t N(t)\, dt = \text{constant} \tag{4}$$

then the Lagrangian is

$$L \equiv -\int_a^b [N(t) \ln N(t) + \mu \int_0^t N(t)\, dt]\, dt$$

$$= -\int_a^b \left[\frac{du}{dt} \ln \frac{du}{dt} + \mu\, (u(t) - u(o))\right] dt, \tag{5}$$

where $du/dt = N(t)$. (6)

Using Eüler-Lagrange equation, we get

$$\mu = \frac{d}{dt}\left(1 + \ln \frac{du}{dt}\right). \tag{7}$$

Integrating and using (6), we get

$$N(t) = N(o) \exp(\mu t), \tag{8}$$

which is the law of population growth given by Malthus (1798).

3. *Logistic Law of Population Growth*

Here we use Kapur's measure of entropy viz.

$$S = -\int_a^b [N(t) \ln N(t) - \frac{1}{\nu}(c - \nu N(t)) \ln (c - \nu N(t))]\, dt, \tag{9}$$

for which the basic motivation was provided by the need to derive Bose-Einstein and Femi-Dirac distributions of statistical mechanics from the maximum-entropy principle [Kapur (1972, 1983a, b)].

This can be used when there is reason to believe that the population size is bounded. Maximizing S subject to constraint (4), we get

$$\mu - \frac{d}{dt}\left[\ln \frac{N(t)}{c - \upsilon N(t)}\right] = 0$$

or

$$\frac{dN}{dt} = \frac{\mu}{c} N(t)\, (c - \upsilon N(t)) \tag{10}$$

which is the logistic law of growth.

4. *Other Possible Laws of Population Growth*

If we use the φ-entropy given by

$$S = -\int_a^b \varphi(N(t))\, dt, \tag{11}$$

where $\varphi(\,.\,)$ is a convex twice-differentiable function and use the constraint (4), we get

$$\varphi'(N(t)) - \varphi'(N(o)) = \mu t. \tag{12}$$

If $\varphi(x) = x \ln x$, we get Malthus law of population growth, i.e., (3).

If $\varphi(x) = x \ln x - \frac{1}{\upsilon}(c - \upsilon x) \ln (c - \upsilon x)$, we get logistic law of a growth, i.e., (10).

If $\varphi(x) = x^2$, we get the linear law of growth

$$N(t) = N(o) + \tfrac{1}{2}\mu t. \tag{13}$$

If $\varphi(x)=1/x$, we get the squared-harmonic law of growth

$$(N(t))^{-2}-(N(o))^{-2}=-\mu t. \tag{14}$$

If $\varphi(x)=x^{\alpha}\,(\alpha>1)$, we get the law of growth

$$N(t)=[(N(o))^{\alpha-1}+\frac{\mu}{\alpha}t]^{\frac{1}{\alpha-1}} \tag{15}$$

corresponding to the measures of entropy given by Renyi (1961) and Havarda and Charvat (1967).

References

1. J. H. Havarda and F. Charvat (1967) Kybernatics *3*, 30-35.
2. E. T. Jaynes (1957) Physical Reviews *106*, 620-630, *108*, 171-197.
3. J. N. Kapur (1972) Journ. Ind. Soc. Ag. Stat. *24*, 47-66.
4. J. N. Kapur (1983a) Journ. Inf. Opt. Sci. *4*, 207-232.
5. J. N. Kapur (1983b) Journ. Math. Phy. Sci. 17, 103-156.
6. T. R. Malthus (1798) *An essay on populations*, St. Paul's, London.
7. R. Pearl and I. J. Reed (1920) Proc. Nat. Acad. Sci. USA, *6*, 273-278.
8. A. Renyi (1961) Proc. 4th Berkeley Symp. Math. Stat. and Prob. *1*, 547-56.
9. C. E. Shannon (1948) Bell. System Tech. J. *27*, 379-423.
10. P. F. Verhulst (1845) Nuov Mon Acad. Roy. Brussels *18*, 1-20.
11. V. Volterra (1928) In *Lecture Notes on Biomathematics 22*, 11-17, Springer-Verlag.