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FLOW OF SECOND-ORDER FLUIDS BETWEEN ECCENTRIC CYLINDERS

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Rheometric flow system which makes use of the steady flow of an incompressible second order fluid between two eccentric cylinder rotating with the same angular velocity about their respective axes has been considered. Relation between the forces on inner cylinder and the material coefficients has been obtained.

1. *Introduction*

Many methods have been devised to determine the material coefficients of fluid models. Maxwell Orthogonal Rheometer and Kepes Balance Rheometer are such instruments where certain force or couple measurements could be immediately converted into relevant material coefficient data. Wood (1957) and Kamal (1966) have considered rheometric systems for viscous fluids where inner cylinder of two eccentric cylinders is rotating and the outer one is stationary. Walter (1970), Abbott and Walter (1970, 1970) have considered such rheometric systems for Walter's liquid. Here we consider rheometric system which makes use of the steady flow of an incompressible second-order fluid between two eccentric cylinders which rotate about their respective axes with the same angular velocity, and relate the material coefficients to the forces on the inner cylinder.

The constitutive equation of an incompressible second-order fluid as proposed by Coleman and Noll (1960) is

$$\left.\begin{aligned} \tau_{ij} &= -p\delta_{ij} + \mu_1 D_{ij} + \mu_2 E_{ij} + \mu_3 D_{ik} D^k_j, \\ D_{ij} &= v_{i,j} + v_{j,i}\,;\; E_{ij} = a_{i,j} + a_{j,i} + 2v^m{}_{,i}\, v_{m,j} \end{aligned}\right\} \tag{1}$$

where v_i and a_i are the components of velocity and acceleration respectively and τ_{ij} is the stress tensor. p is an indeterminate pressure and μ_1, μ_2, μ_3 are material coefficients.

2. *Basic Theory*

The steady flow of an incompressible second-order fluid between two eccentric cylinders of radii R_1 and R_2 $(R_1 < R_2)$ which rotate about their respective axes with the same angular velocity Ω will be considered.

All physical quantities will be referred to cylindrical polar coordinates (r, θ, z), the z-axis being along the axis of the inner cylinder. The distance between the axes of the cylinders will be denoted by a, In most of what follows we shall work to first order in a.

If u, v, and w are the velocity components in r, θ and z directions, respectively, the relevent boundary conditions to order a are

$$\left.\begin{array}{llll} u=0, & v=\Omega R_1, & w=0 & (ON)\ r=R_1 \\ u=\Omega a\cos\theta, & v=\Omega R_2, & w=0 & (ON)\ r=R_2+a\sin\theta \end{array}\right\} \tag{2}$$

These conditions suggest a velocity distribution of the form

$$u=\Omega aF(x)e^{i\theta},\ v=\Omega[xR_1+ia(xF)'e^{i\theta}],\ w=0 \tag{3}$$

where $x=r/R_1$ and a dash denotes d/dx. The terms of order a^2 have been neglected in obtaining (3). The form (3) satisfies the equation of continuity identically. Writing the pressure distribution p in the form

$$p(x,\theta)=\rho R_1^2\Omega^2\left[\frac{x^2}{2}+\frac{ia}{R_1}p_1(x)e^{i\theta}\right]+\text{constant} \tag{4}$$

the equations of motion for an incompressible second-order fluid take the form

$$F+2xF'=p_1'+\frac{1}{\alpha^2}\left(F''+\frac{3}{x}F'\right) \tag{5}$$

$$F-xF'=\frac{1}{x}p_1+\frac{1}{\alpha^2}\left(4F''+xF'''\right) \tag{6}$$

where ρ is the density of fluid and

$$\alpha^2=-\frac{i\rho\Omega R_1^2}{\mu_1+i\mu_2\Omega} \tag{7}$$

$F(x)$ and $p_1(x)$ may be complex and the real part will be implied throughout. Eliminating $p_1(x)$ between (5) and (6), we obtain the equation for F as

$$(D^3-4D+\alpha^2x^2D)F=K \tag{8}$$

where K is a constant to be determined and $D\equiv x\dfrac{d}{dx}$. This equation has to be solved subject to the conditions

$$F(1)=0,\ F'(1)=0,\ F(\lambda)=1,\ F'(\lambda)=0 \tag{9}$$

where $\lambda=(R_2+a\sin\theta)/R_1$.

The equation (8) and the boundary conditions (9) are similar to those obtained by Abbott and Walter (1970) for the corresponding flow problem for Walter's liquid. In equation (8) above α^2 involves material coefficients defining an incompressible second-order fluid and is of interest to express these coefficients in terms of measurable forces on the inner cylinder.

The solution of the equation (8) can be written in terms of Bessel functions but is likely to be too complicated to be useful for interpreting experimental results. However, in most cases the inertial effects are small and we write the solution of (8) by expanding F as a powers series in α^2 in the form.

$$F=F_0+\alpha^2F_2+\cdots,\quad K=K_0+\alpha^2K_2+\cdots. \tag{10}$$

The conditions (9) become

$$\begin{aligned}&F_{2n}(1)=0,\quad F'_{2n}(1)=0,\quad F_0(\lambda)=1.\\&F_{2n+2}(\lambda)=0,\quad F'_{2n}(\lambda)=0\quad(n=0,1,2,\cdots).\end{aligned} \tag{11}$$

Substituting (10) in (8) and equating the coefficients of like powers of α^2, we get a system of equations for F_{2n} and K_{2n}. For $n=0$ and 1, there are given as

$$\begin{aligned}&F_0=\frac{1}{M}[(x^2-1)(1+\frac{\lambda^2}{x^2})-(\lambda^2+1)\ln x^2],\ K_0=\frac{8}{M}(1+\lambda^2),\\&F_2=Ax^2+\frac{B}{x^2}-\frac{K_2}{4}\ln x+C+\frac{1}{4M}\Big[2\lambda^2\ln x+(1+\lambda^2)x^2\ln x-\frac{1}{6}x^4\Big]\\&K_2=\frac{2}{3M^2}[12\lambda^2(\lambda^2+1)\ln\lambda-(\lambda^2-1)(\lambda^4+10\lambda^2+1)]\end{aligned} \tag{12}$$

where

$$\left.\begin{aligned}&M=2(\lambda^2+1)\ln\lambda-2(\lambda^2-1)\\&A=\frac{\ln\lambda}{12(\lambda^2-1)M^2}[1+5\lambda^6-3\lambda^2(1+\lambda^2)(1+2\lambda^2\ln\lambda)\\&B=\frac{\lambda^2}{12(\lambda^2-1)M^2}[(\lambda^2+1)(\lambda^2-1)^2\\&\qquad+2\lambda^2(\lambda^2-1)\ln\lambda-6\lambda^2(\lambda^2+1)(\ln\lambda)^2]\\&C=\frac{1}{3M^2}[(3\lambda^4+\lambda^2-2)\ln\lambda-(\lambda^2-1)(\lambda^4+\lambda^2-1)]\end{aligned}\right\} \tag{13}$$

Let X and Y be the forces on the inner cylinder in the $\theta=0$ and $\theta=\frac{\pi}{2}$ directions respectively. These are given by

$$X-iY=LR_1\int_0^{2\pi}(\tau_{rr}-i\tau_{r\theta})e^{-i\theta}\,d\theta \tag{14}$$

where τ_{rr} and $\tau_{r\theta}$ are the stresses evaluated at $x=1$, and L is the length of the column of fluid. Substituting the expressions for the stresses, we obtain

$$\begin{aligned} X-iY &= -a\pi\Omega L(\mu_1+i\mu_2\Omega)[F'''(1)+3F''(1)] \\ &= -a\pi\Omega L(\mu_1+i\mu_2\Omega)(K_0+\alpha^2K_2) \end{aligned} \tag{15}$$

where terms of order α^4 have be neglected. Separating the real and imaginary parts in (15), we obtain

$$\left.\begin{aligned} X &= \frac{4a\pi\Omega L\mu_1(\lambda^2+1)}{(\lambda^2-1)-(\lambda^2+1)ln\lambda}, \\ Y &= a\pi\Omega^2 L\left[\frac{48\mu_2}{ln\lambda-\frac{\lambda^2-1}{\lambda^2+1}}-\frac{\rho R_1^2}{3M}\cdot\frac{4\lambda^2 l_n\lambda-\lambda^2+1}{ln\lambda-\frac{\lambda^2-1}{\lambda^2+1}}-\frac{4\rho R_1^2\lambda^2}{3M}\right] \end{aligned}\right\} \tag{16}$$

Thus, to the order α^2, the force in the $\theta=0$ direction is linearly proportional to μ_1 while that in the $\theta=\frac{\pi}{2}$ direction may exist due to geometrical and flow parameters only.

Further, it can be shown that the terms of the order a^2 will not affect the expression for the force X-iY. Thus, the linear effects will not manifest themselves until terms of order a^3 are considered.

3. *On-line Control*

Rheometers which could provide 'on line' control in industrial processes should have to be located ideally in the process without affecting the basic flow. In the case of the eccentric-cylinder rheometer one would therefore envisage a flow along the cylinders in addition to the flow created by the rheometer itself. However, this superimposed flow will modify the forces X and Y. To demonstrate this, we consider a constant pressure gradient g in the z-direction in addition to the situation considered earlier. We assume a velocity and pressure distribution of the form

$$\left.\begin{aligned} &u=a\Omega F(x)e^{i\theta},\quad v=\Omega[xR_1+ia(xF)']e^{i\theta} \\ &w=\Omega\left[R_1\varphi(x)+iaG(x)e^{i\theta}\right],\quad p=gz+\rho R_1^2\Omega^2\left[\tfrac{1}{2}x^2+p_2(x)+\frac{ia}{R_1}p_1e^{i\theta}\right] \end{aligned}\right\} \tag{17}$$

The boundary conditions are

$$\left.\begin{aligned} &F=0,\ F'=0,\ \varphi=0,\ G=0 \text{ at } x=1 \\ &F=1,\ F'=0,\ \varphi=0,\ G=\varphi' \text{ at } x=\lambda \end{aligned}\right\} \tag{18}$$

Substituting (17) in the equations of motion and equating the terms dependent on z, θ and independent of these to zero separately, we get

$$\frac{1}{x}(x\varphi')'=\frac{gR_1}{\Omega\mu_1}\equiv N,\quad p'_2=2\beta E\left(2\varphi''+\frac{1}{x}\varphi'\right)\varphi' \tag{19}$$

$$p_1(x)=xF-x^2F'-\frac{1}{\alpha^2}(x^2F'''+4xF'')+(2\beta+\gamma)E(NG+\varphi'G') \tag{20}$$

$$3xF'+x^2F''+\frac{1}{\alpha_2}\left(x^2F^{iv}+6xF'''+3F''-\frac{3}{x}F'\right)$$
$$+\beta E\left(\frac{1}{x^2}\varphi'G-\varphi''G'\right)-(\beta+\gamma)E\left(\varphi'G''+\varphi''G'+NG'-\frac{1}{x}\varphi'G\right)=0 \tag{21}$$

$$G+\frac{1}{\alpha_2}\left(G''+\frac{1}{x}G'-\frac{1}{x^2}G\right)-\varphi'F+2\beta E\varphi''F'+(\beta-\gamma)E(xF')'\frac{1}{x}\varphi'=0 \tag{22}$$

where

$$E=\frac{\mu_1}{\rho\Omega R_1^2},\quad \beta=\frac{\mu_2\Omega}{\mu_1},\quad \gamma=\frac{\mu_3\Omega}{\mu_1}.$$

Here we have used the equation (20) in obtaining (21). The equations (19) give

$$\varphi(x)=\frac{N(\lambda^2-1)}{4}\left[\frac{x^2-1}{\lambda^2-1}-\frac{\ln x}{\ln\lambda}\right] \tag{23}$$

$$p_2(x)=\frac{\beta N^2E}{8}\left[6x^2-4(\lambda^2-1)\frac{\ln x}{\ln\lambda}+\frac{1}{2x^2}\left(\frac{\lambda^2-1}{\ln\lambda}\right)^2\right]. \tag{24}$$

The coupled equations (21) and (22) may now be solved for F and G subject to the conditions (18). The equations are complicated to be solved. However, assuming a small pressure gradient and expanding F and G as power series in N as

$$F=\sum_{j=0}^{\infty}N^jF_{(j)},\quad G=\sum_{j=0}^{\infty}N^jG_{(j)}.$$

We can easily conclude that $G_{(0)}=0$ and $F_{(1)}=0$. This implies that F is not affected unless the terms of order N^2 are considered.

The complex force $X-iY$ is given by

$$X-iY=a\pi\Omega L(\mu_1+i\mu_2\Omega)[F'''(1)+3F''(1)]$$
$$+ia\pi\Omega^2L(2\mu_2-\mu_3)\varphi'(1)\;G'(1). \tag{25}$$

We observe that the forces x and y depend upon the material coefficient μ_3 also in this case. Since $G(x)$ is complex in general, both X and Y are affected by the superimposed flow. However, if the applied pressure gradient g is so small that its square is negligible, then (25) will be identical to (15), and the forces will not be affected by the superimposed flow.

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