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EFFECT OF INJECTION ON THE HEAT TRANSFER IN FLOW OF A SECOND-ORDER FLUID

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ABSTRACT

The flow and the heat transfer of a second-order fluid confined between two disks, one rotating and the other at rest, have been discussed. The rotating disk is maintained at a temperature higher than that of the stationary disk. There is a small amount of fluid injection at the stationary disk. The case when rotating disk is insulated has also been discussed. The equations have been solved by expanding the velocity components and the temperature in the powers of Reynolds number.

The results reveal that the effects of injection of the fluid and non-Newtonian terms in the constitutive equation are to reduce the rate of heat transfer from both disks. It has been found in thermometric case (when the rotating disk is insulated) that the effect of the injection of the fluid is to increase the temperature of the rotating disk but the effect of non-Newtonian terms is contrary to it.

1. *Introduction*

The constitutive equation of an incompressible second-order fluid has been suggested by Coleman and Noll (1) as :

$$\tau = -p\delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)ik} A_{(1)kj}, \tag{1}$$

where

$$A_{(1)ij} = v_{i,j} + v_{j,i}$$
$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i} v_{m,j} \tag{2}$$

τ_{ij} is the stress tensor, v_i and A_i are the velocity and the acceleration vectors, respectively ; μ_1, μ_2 and μ_3 are the material constants and p is the hydrostatic pressure and a comma ',' denotes covariant dicerentiation with respect to index following it. Markovitz and Coleman (2) have shown that 5.4 percent solution of poly-iso-butylene in cetane at 30°C behaves as a second order fluid with material constants as $\rho=0.733$ g/cm, $\mu_1=18.5$ poises, $\mu_2=-0.2$ g/cm and $\mu_3=1.0$ g/cm.

In this paper we have discussed the flow and the rate of heat transfer of a second-order fluid governed by the constitutive equations (1) and (2) when it is confined between two disks, one of which rotates with a constant angular velocity Ω and the other is at rest. The disk which is stationary is porous and a small amount of fluid is injected through it. Two cases have been discussed :

1. When the rotating and stationary disks are maintained at constant temperatures T_1 and T_0 respectively ($T_1 > T_0$).
2. When the stationary disk is maintained at a constant temperature T_0 and the rotating disk is insulated. Assuming Reynolds number R to be small, the solution of the differential equations has been sought by expanding all the entities in powers of R. The interaction of the fluid injection and non-Newtonian terms in the constitutive equation has been discussed. The graph of the Nusselt number at the stationary disk against a non-dimensional injection number has been plotted by taking three values of non-dimensional temperature difference between two disks for the Newtonian as well as for second order fluids. This problem in the absence of injection has been discussed by Srivastava (4).

2. *Equations of Motion and Heat Transfer*

The equation of continuity and Cauchy's equation of motion for incompressible fluid for a steady flow are given by qfmm

$$v_{i,\,i}=0, \tag{3}$$

$$\rho v_j v_{i,\,j}=\tau_{ij,\,j},$$

where ρ is the density of the fluid.

The equation for heat transfer is given by

$$\rho c_p v_i \frac{\partial T}{\partial x^i}=kT_{,ii}+\Phi, \tag{5}$$

where T is the temperature at any point of the fluid, C_b is the specific heat at a constant pressure, x^i is a space coordinate, k is the thermal conductivity and ϕ is the dissipation functiion given by

$$\Phi=\tfrac{1}{2}\,A_{(1)ij}\,\tau_{ij}. \tag{6}$$

Consider the motion of a second-order fluid governed by equations (1) and (2) between two infinite disks, one of which ($z=o$) is at rest and the other ($z=d$) is rotating with a constant angular velocity about an axis ($r=o$) perpendicular to its own plane. The rotating disk is maintained at a constant temperature T_1 and the stationary disk is maintained at another constant temperature T_0 ($T_1 > T_0$). Taking u, v and w to be the velocity components in the directions of r, θ and z respectively, in cylindrical polar dinates, the boundary conditions of this problem are given by :

$$u=0,\ v=0,\ w=w_0,\ T=T_0 \text{ at } z=0, \tag{7}$$

$$u=0,\ v=r\Omega,\ w=0,\ T=T_1 \text{ at } z=d. \tag{8}$$

Boundary conditions suggest the following form of the velocity components, temperature and pressure.

$$u=r\Omega F'(\eta),\ v=r\Omega G(\eta),\ w=-2d\Omega F(\eta), \tag{9}$$

$$T=T_0+(\mu_1\Omega/ec_p)[\phi(\eta)+(r^2/d^2)\psi(\eta)], \tag{10}$$

$$p=\mu_1\Omega[-p_1+(r^2/d^2)\{(2A+B)(F''^2+G'^2)+\lambda\}], \tag{11}$$

where $\eta=z/d$, $A=\mu_2\Omega/\mu_1$, $B=\mu_3\Omega/\mu_1$, a prime denotes differentiation with respect to η and λ is a constant to be determined by the boundary conditions. This form of velocity components satisfies the equation of continuity (3) identically.

Substituting the velocity components from (9), pressure from (11) and stress components from (1) and (2), the equations of motion in the directions of r, θ and z respectively are :

$$R(F'^2-G^2-2FF'')=F'''-2\alpha R(F''^2+2G'^2+FF^{iv})$$
$$-\beta R(F''^2+3G'^2+2F'F''')-2\lambda, \tag{12}$$

$$2R(F'G-FG')=G''+2\alpha R(F''G'-FG'')+2\beta R(F''G'-F'G''), \tag{13}$$

$$4RFF'=p_1'-2F''+4\alpha R(11F'F''+FF''')+28\beta RF'F'', \tag{14}$$

where $\alpha=\mu_2/ed^2$, $\beta=\mu_3/ed^2$ and $R=e\Omega d^2/\mu_1$.

Substituting velocity components from (9), temperature from (10) and stress components from (1) and (2) in the equations (5) and (6), and equating the terms independent of r and coefficient of r^2 on both sides of the equations (5), we get

$$(RP)^{-1}(4\psi+\phi'')+12F'^2+2F\phi'-24\alpha FF'F''-24(\alpha+\beta)F'^2=0 \tag{15}$$

$$(RP)^{-1}\psi''+F''^2+G'^2-2(\psi F'-F\psi')-2\alpha F(F''F'''+G'G'')$$
$$-(2\alpha+3\beta)F'^2(F''^2+G'^2)=0, \tag{16}$$

where $P=\mu_1 c_p/k$ is the Prandtl number.

3. *Solution of the Equations*

Assuming the Reynolds number R to be small we develop a regular perturbation scheme by expanding F, G, ϕ, ψ and λ in powers of R. Neglecting R^2 and its higher powers, we write

$$F=F_0+RF_1,\ G=G_0+RG$$
$$\phi=\phi_0+R\phi_1,\ \psi=\psi_0+R\psi_1$$
$$\lambda=\lambda_0+R\lambda_1. \tag{17}$$

The boundary conditions (7) and (8) can be written as :

$$F_0=-S,\ F'_0=0,\ G_0=0 \text{ at } \eta=0,$$
$$F_0=0,\ F'_0=0,\ G_0=1 \text{ at } \eta=1 \tag{18}$$

$$\left.\begin{array}{l}\phi_0=0,\ \psi_0=0 \text{ at } \eta=0,\\ \phi_0=m,\ \psi_0=0 \text{ at } \eta=1.\end{array}\right\} \tag{19}$$

$$F_1=F'_1=G_1=0 \text{ at } \eta=0 \text{ and } \eta=1, \tag{20}$$

$$\phi_1=\psi_1=0 \text{ at } \eta=0 \text{ and } \eta=1, \tag{21}$$

where $S=w_0/2d\Omega$ and $m=(T_1-T_0)\rho c_p\,\Omega/\mu_1$.

Substituting (17) in the equations (12), (13) and (15), (16) and equating the terms independent of R and coefficient of R on both sides of these equations, we get two sets of linear differential equations in $(F_0, G_0, \phi_0, \psi_0, \lambda_0)$ and $(F_1, G_1, \phi_1, \psi_1, \lambda_1)$. Solving them under boundary conditions given by (18) to (21), we get the solutions for

$(F_0, G_0, \phi_0, \psi_0, \lambda_0)$ and $(F_1, G_1, \phi_1, \psi_1, \lambda_1)$ as :

$$F_0=-S(1-3\eta^2+2\eta^3),\ G_0=\eta,\ \lambda_0=-6S, \tag{22}$$

$$\phi_0=m\eta,\ \psi_0=0, \tag{23}$$

$$F_1=\frac{1}{420}\eta^2(1-\eta)^2[12s^2(-22+8\eta+3\eta^2-2\eta^3)$$
$$-7(2+\eta)-252(4\alpha+3\beta)s^2(1-2\eta)],$$

$$G_1=-\frac{s}{10}\eta(1-\eta)[11+\eta+\eta^2-4\eta^3-20(\alpha+\beta)(1-2\eta)],$$

$$\lambda_1=\frac{1}{140}\Big[21-216s^2-56\alpha(18s^2+5)+42\beta(12s^2-5)],\Big] \tag{24}$$

$$\phi_1=\frac{P\eta(1-\eta)}{30}\Big[12s^2(21+21\eta+11\eta^2-64\eta^3+32\eta^4)$$
$$+3sm(-7+3\eta+3\eta^2-2\eta^3)+5(1+\eta-\eta^2)$$
$$-48\alpha s^3(5+5\eta+81\eta^2-39\eta^3-225\eta^4+279\eta^5$$
$$-93\eta^6)-216\beta s^3(3+3\eta+2\eta^2+2\eta^3-31\eta^4+36\eta^5$$
$$-12\eta^6)-2(2\alpha+3\beta)s(3+3\eta-2\eta^3+\eta^4)].$$

$$\psi_1=\frac{P\eta(1-\eta)}{10}[60s^2(1-2\eta+2\eta^2)+5$$
$$-48\alpha s^3(7-8\eta-13\eta^2+32\eta^3-16\eta^4)$$
$$-108\beta s^3(1+\eta-9\eta^2+16\eta^3-8\eta^4)$$
$$-5(2\alpha+3\beta)(1+\eta-\eta^2)]. \tag{25}$$

4. *Thermometric Case*

Here we discuss the above problem with the following boundary conditions on the temperature.

$$T=T_0 \text{ at } \eta=0,$$
$$\frac{\partial T}{\partial \eta}=0 \text{ at } \eta=1. \tag{26}$$

The boundary conditions on the velocity components remain unchanged. Hence, for this problem the set of differential equations for $(F_0, G_0, \phi_0, \psi_0, \lambda_0)$ and $(F_1, G_1, \phi, \psi_1, \lambda_1)$ and the boundary conditions given by (18) and (20) remain unchanged, but the boundary conditions given by (19) and (21) become :

$$\phi=0,\ \psi_0=0 \text{ at } \eta=0,$$
$$\phi_0'=0,\ \psi_0'=0 \text{ at } \eta=1, \tag{27}$$
$$\phi_1=0,\ \psi_1=0 \text{ at } \eta=0,$$
$$\phi_1'=0,\ \psi_1'=0 \text{ at } \eta=1. \tag{28}$$

Under the above boundary conditions (27), (28), the solution for $\phi_0, \psi_0, \phi_1, \psi_1$ are given by

$$\phi_0=m,\ \psi_0=0, \tag{29}$$

$$\begin{aligned}\phi_1=\tfrac{1}{30} P\eta[&12s^2(72-20\eta^2-75\eta^3+96\eta^4\\&-32\eta^5)+5(8-4\eta^2+\eta^3)-4\alpha s\{36s^2(10+24\eta^2\\&-40\eta^3-62\eta^4+168\eta^5-124\eta^6+31\eta^7)\\&+(21-10\eta^2+3\eta^4-\eta^5)\}-6\beta s\{36s^2(9\\&-2\eta^2-33\eta^4+67\eta^5-48\eta^6+12\eta^7)\\&+(21+10\eta^2+3\eta^4-\eta^5)\}],\end{aligned}$$

$$\begin{aligned}\psi_1=\tfrac{1}{60} P\eta[&360s^2(2-3\eta+4\eta^2-2\eta^3)\\&+30(2-\eta)+12\alpha\{24s^3(-9+15\eta+5\eta^2\\&-45\eta^3+48\eta^4-16\eta^5)+5s(-2+2\eta^2\\&-\eta^3)\}+18\beta\{36s^3(-2+10\eta^2-25\eta^3\\&+24\eta^4-8\eta^5)+5s(-2+2\eta^2-\eta^3)\}].\end{aligned} \tag{30}$$

5. *Discussion*

The rate of heat flux from the stationary disk to the fluid is given by

$$[q(r)]_{z=0}=-k\left(\frac{\partial T}{\partial z}\right)_{z=0} \tag{31}$$

Neglecting the edge effects, the expression for the rate of heat transfer from a circular portion of radius a of the disk at $z=o$ is given by

$$\begin{aligned}q_0&=\frac{1}{\pi a^2}\int_0^a -\frac{2\pi r}{d}\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0} dr\\&=\frac{\mu_1\Omega}{\rho c_p}\left[m+\frac{RP}{30}\left\{\left(252s^2+5\right)-21sm\right.\right.\\&\qquad\left.-12\alpha s(20s^2+1)-18\beta s(36s^2+1)\right\}\end{aligned}$$

$$+\frac{RP}{20}X^2\{5(12s^2+1)-2\alpha s(168s^2+5)$$
$$-3\beta s(36s^2+5)\}\Big] \quad (32)$$

where $\chi=a/d$.

The graph of Nusselt number $N=-q_0d/k(T_1-T_0)$ has been plotted against S for $m=10$, 15 and 20 in Fig. 1 for a second order fluid by taking $\alpha=-0.03$ and $\beta=0.15$ and just for comparison sake the corresponding curves for Newtonian fluid ($\alpha=0$, $\beta=0$) have been shown by broken lines. Here α is taken as negative and, β as positive and numerically five times α which is consistent with the values of μ_2 and μ_3 given by Markovitz and Coleman (2).

The graph shows that the rate of heat transfer decreases with the increase of injection parameter S. Further, this rate of decrease becomes more and more pronounced as the parameter m increases. This shows that the fluid injection is more effective in reducing the rate of heat transfer when the temperature difference between the disks increases.

The effect of second-order terms in the constitutive equation (1) is to reduce further the rate of heat transfer. Neglecting the edge effects, the rate of heat transfer from the circular portion of radius α of the rotating disk ($z=d$) is given by

$$q_1=\frac{\mu_1\Omega}{\rho c_p}\Big[m+\frac{RP}{30}\{-(252s^2+5)+9.s.m$$
$$+12\alpha 5(52s^2+1)+18\beta s(36s^2+1)\}$$
$$+\frac{RP}{20}x^2\{-5(12s^2+1)+2\alpha s(48s^2+5)$$
$$+3\beta s(36s^2+5)\}\Big]. \quad (33)$$

The graph of q_1 has not been plotted but it can be concluded from the expression that the effects of injectiton and non-Newtonian terms are similar as those of q_0. These results have been supported by many workers (see Schlichting (3)) that by injecting a small amount of fluid the rate of heat transfer can be reduced. In many situations fluid is injected to cool a rotating system. Our results suggest that the cooling system will be more effective if the fluid chosen is a non-Newtonian one.

In thermometric case, the temperature T^* of the rotating disk can be obtained by substituting the expressions for ϕ_0, ψ_0, ϕ_1 and ψ_1 from (29) and (30) in (10). It is give by

$$T^*=T_0+\frac{\mu_1\Omega}{\rho C_p}\Big[\frac{RP}{30}\{(492s^2+25)-4\alpha s(252s^2+13)$$
$$-6\beta s(180s^2+13)\}+\frac{RP}{20}x^2\{s(12s^3+1)$$
$$-2\alpha s(48s^2+5)-3\beta s(36s^2+5)\}\Big]. \quad (34)$$

The values of $(T^*-T_0)\rho C\rho/\mu_1$ have been given in table 1 for $S=0.05$, 0.10, 0.15 and for $\alpha=0$, $\beta=0$, $\alpha=-0.01$, $\beta=0.05$, $\alpha=-0.02$, $\beta=0.10$, $\alpha=-0.03$, $\beta=0.15$. The table shows that the effect of injection is to increase the temperature but the effects of increase of non-Newtonian parameters α and β is to decrease the temperature. This is in agreement with the results of Srivastava (4) who showed that the effect of non-Newtonian terms is to reduce the temperature of the disk.

TABLE—1

S	$\alpha=0$, $\beta=0$,	$\alpha=-0.01$ $\beta=0.05$	$\alpha=-0.02$ $\beta=0.10$	$\alpha=-0.03$ $\beta=0.15$
0.05	57.13	56.757	56.384	56.012
0.10	63.52	62.721	61.923	61.124
0.15	74.17	72.840	71.510	70.180

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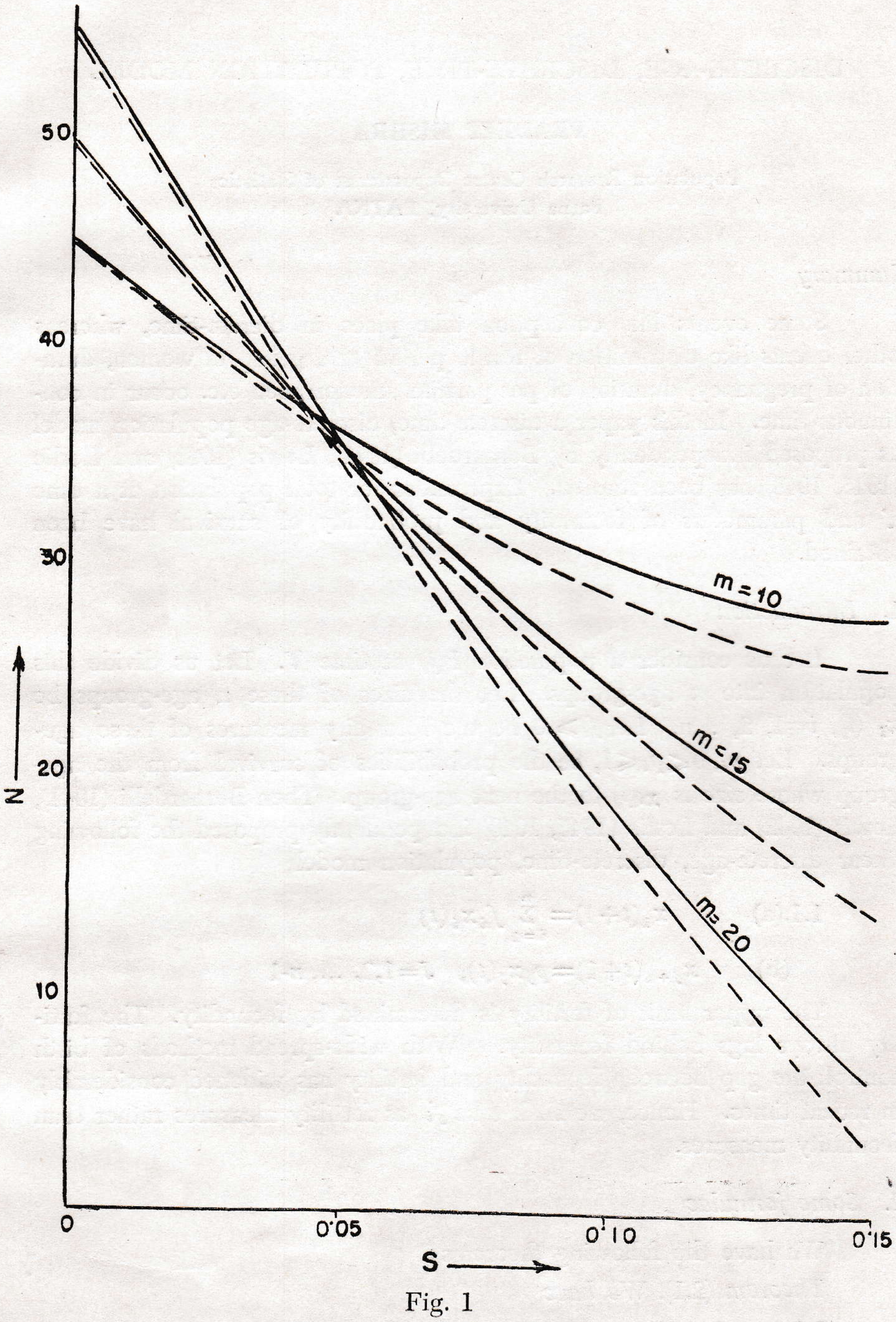

Fig. 1

The graph of N against S

——— NEWTONIAN FLUID

- - - - - - NON NEWTONIAN FLUID