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DISCRETE-AGE, DISCRETE-TIME, POPULATION MODEL

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Summary

Some events like conception take place in dicrete-time, whereas other events like distribution of fertile period of a group of women, duration of pregnancy, duration of postpartum amenorrhoea etc. occur in continuous time. In this paper a discrete-time, discrete-age population model as proposed independently by Bernardelli (1941), Lewis (1942) and Leslie (1945, 1948) has been studied. Expressions for total population at a time 't' and parameters of fecundity and probability of survival have been obtained.

1. *Introduction*

Let us consider a population $P(t)$ at time 't'. Let us divide this population into n age-groups. Let the sizes of these n age-groups be $x_i\ (t)$, $i=1, 2, \ldots n$. Let $f_i \geqslant 0$ be the fecundity measures of these age-groups. Let p_i, $0 \leqslant p_i \leqslant 1$, be the probabilities of survival from the age-group whose size is $x_i(t)$ to the next age-group. Then Bernardelli (1941), Lewis (1942) and Leslie (1945, 1948) independently proposed the following linear discrete-age, discrete-time, population model

1.1.(a) $$x_1(t+1)=\sum_{i=1}^{n} f_i x_i(t)$$

(b) $$x_{j+1}(t+1)=p_j x_j(t), \quad J=1,2, \ldots, n\text{-}1$$

The upper limit of fertility is determined by fecundity. The fertility always lags behind fecundity. With wide-spread methods of birth control, the gap between fecundity and fertility has widened considerably in recent times. Hence, we shall take f_i as fertility measures rather than fecundity measures.

2. *Some formulae*

We have the following theorems :

Theorem (2.1). *We have*

2.1. $$x_1(t)=f_1x_1(t-1)+p_1f_2x_1(t-2)+p_1p_2f_3x_1(t-3) +\cdots\cdots+p_1p_2\cdots p_{n-1}f_nx_1(t-n).$$

Proof. Repeated use of (1.1)b in (1.1)a yields (2.1).

Note (2.1). In (2.1), we have reduced all the terms on the right-hand side of (1.1)a to the base x_1.

Theorem (2.2). *Let us put*

2.2. (a) $A_0=1$, (b) $A_{-r}=0$,

(c) $A_r=f_1A_{r-1}+p_1f_2A_{r-2}+p_1p_2f_3A_{r-3}+\cdots\cdots+p_1p_2\cdots p_{r-1}f_r\,;\quad 1\leqslant r\leqslant n.$

Then

2.3. $x_1(t)=A_rx_1(t-r)+(p_1f_2A_{r-1}+p_1p_2f_3A_{r-2}+\cdots+p_1p_2\cdots p_rf_{r+1})x_1(t-r-1)+\cdots\cdots+(p_1\cdots p_{n-r}f_{n-r}A_{r-1}+\cdots+p_1\cdots p_{n-1}f_n)x_1(t-n)+(p_1\cdots p_{n-r+1}+f_{n-r+2}A_{r-1}+\cdots+p_1\cdots p_{n-1}f_nA_1)x_1(t\text{-}n-1)+\cdots+p_1p_2\cdots p_{n-1}f_nA_{r-1}x_1(t-n-r+1).$

Proof: from (2.1), we can easily write

2.4.(a) $x_1(t-1)=f_1x_1(t-2)+p_1f_2x_1(t-3)+\cdots+p_1\cdots\cdots p_{n-1}f_nx_1(t-n-1)$

(b) $x_1(t-2)=f_1x_1(t-3)+\cdots+p_1\cdots p_{n-1}f_nx_1(t-n-2),$

(c) $x_1(t-3)=f_1x_1(t-4)+\cdots+p_1\cdots p_{n-1}f_nx_1(t-n-3),$

(d) $x_1(t-r)=f_1x_1(t-r-1)+\cdots+p_1\cdots p_{n-1}f_nx_1(t-n-r).$

Substituting from (2.4)a in (2.1), we obtain

2.5 (a) $x_1(t)=A_2x_1(t-2)+(p_1f_2A_1+p_1p_2f_3)x_1(t-3)+(p_1p_2f_3A_1+p_1p_2p_3f_4)x_1(t-4)+\cdots+p_1\cdots p_{n-1}f_nA_1x_1(t-n-1).$

Substituting from (2.4)b in (2.5)a, we get

(b) $x_1(t)=A_3x_1(t-3)+(p_1f_2A_2+p_1+p_2f_3A_1+p_1p_2p_3f_4)x_1(t-4)+\cdots+p_1p_2\cdots p_{n-1}f_nA_2x_1(t-n-2).$

Proceeding in this way, we can prove (2.3) by induction.

Theorem (2.3). *With the assumption that* (2.3) *holds for* $r>n$, *we have*

(2.6) $x_1(t)=A_rx_1(t-r)+(A_{r+1}-f_1A_r)x_1(t-r-1)+\cdots+(A_n-f_1A_{n-1}-\cdots-p_1\cdots p_{n-r-1}f_{n-r}A_r)x_1(t-n)+(A_{n+1}-f_1A_n-\cdots-P_1\cdots p_nf_{n+1})x_1(t-n-1)+\cdots+(A_{n+r-1}-f_1A_{n+r-2}-\cdots-p_1\cdots p_{n+r-2}f_{n+r-1})x_1(t-n-r+1).$

Proof. Substituting from (2.2) in (2.3) and assuming that (2.2)c holds for $r>n$, we obtain (2.6).

Note (2.1). Though it appears that in the above expression, quantities f_r and p_r occur when $r>n$, but actually they do not occur since they cancel. To avoid this controversy, we will write (2.6) as

$$(2.7)\quad x_1(t)=A_r x_1(t-r)+(A_{r+1}-f_1A_r)x_1(t-r-1) \\ +\cdots+(f_n-f_1A_{n-1}-\cdots-p_1\cdots p_{n-r-1}f_{n-r}A_r)x_1(t-n) \\ +(p_1\cdots p_{n-r+1}f_{n-r+2}A_{r-1}+\cdots+p_1\cdots p_{n-1}f_nA_1)x_1(t-n-1) \\ +\cdots+p_1\cdots p_{n-1}f_nA_{r-1}x_1(t-n-r+1).$$

Note (2.2). Since the number of terms is the same in (2.1), (2.3), (2.6) or (2.7), it is difficult to say which expression is more convenient. It will depend on the situation in which the expression is used.

It may be noted that (2.1) holds when $t\geqslant n$, and (2.3), (2.6) and (2.7) when $t\geqslant n+r-1$, $1\leqslant r\leqslant n$.

In the next section, we will consider the cases

(i) $t<n$, (ii) $n<t<n+r-1$.

3. *Modified equations*

Case 1. Suppose $t=r$, $1\leqslant r<n$. Then (2.1) can be written as

$$(3.1)\quad x_1(t)=f_rx_1(r-1)+p_1f_2x_1(r-2)+\cdots+p_1\cdots p_{r-1}f_rx_1(0) \\ +p_2\cdots p_rf_{r+1}x_2(0)+\cdots+p_{n-r+1}\cdots p_{n-1}f_nx_{n-r+1}(0).$$

Similarly (2.3) can be written as

$$(3{\cdot}2)\quad x_1(t)=A_rx_1(0)+(f_2A_{r-1}+\cdots+p_2\cdots p_rf_{r+1})x_2(0) \\ +(f_3A_{r-1}+\cdots+p_3\cdots p_{r+1}f_{r+2}f_{r+2})x_3(0) \\ +\cdots+(f_{n-r+1}A_{r-1}+\cdots+p_{n-r+1}\cdots p_{n-1}f_n)x_{n-r+1}(0) \\ +(f_{n-r+2}A_{r-1}+\cdots+p_{n-r+2}\cdots p_{n-1}f_nA_1)x_{n-r+2}(0) \\ +\cdots+f_nA_{r-1}x_n(0).$$

Case 2. Let $t=n+q$, $q<r-1$.

In this case (2.1) holds as it is. But we shall modify (2.3). It then assumes the form

$$(3{\cdot}3)\quad x_1(t)=A_rx_1(t-r)+(p_1f_2A_{r-1}+p_1p_2f_3A_{r-2}+\cdots+ \\ p_1p_2\cdots p_rf_{r+1})x_1(t-r-1)+\cdots+ \\ (p_1\cdots p_{n-r}f_{n-r+1}A_{r-1}+\cdots+p_1\cdots p_{n-1}f_nx_1(t-n) \\ +\cdots+(p_1\cdots p_{n-r+q-1}f_{n-r+q}A_{r-1}+\cdots p_1\cdots \\ p_{n-1}f_nA_{q-1})x_1(t)+(p_1\cdots p_{n-r+q}f_{n-r+q+1}A_{r-1} \\ +\cdots+p_1\cdots p_{n-1}f_nA_q)x_1(0)$$

$$+(p_1 \cdots p_{n-r+q+1} f_{n-r+q+2} A_{r-1} + \cdots +$$
$$p_2 \cdots p_{n-1} f_n A_{q+1}) x_2(0) + \cdots + f_n A_{r-1} x_{r-q}(0).$$

Similarly (2.6) can be written as

$$\begin{aligned}
(3.4) \quad x_1(t) = {} & A_r x_1(t-r) + (A_{r+1} - f_1 A_r) x_1(t-r-1) \\
& + \cdots + (A_n - f_1 A_{n-1} \cdots p_1 \cdots p_{n-r-1} f_{n-r} A_r) x_1(t-n) \\
& + \cdots + (A_{n+q-1} - f_1 A_{n+q-2} \cdots p_1 \cdots p_{n+q-2} f_{n+q-1}) x_1(1) \\
& + (A_{n+q} \cdots p_1 \cdots p_{n+q-1} f_{n+q}) x_1(0) \\
& + 1/p_1 (_{n+q+1} \cdots p_1 \cdots p_{n+q} f_{n+q+1}) x_2(0) \\
& + \cdots + \frac{1}{p_1 \cdots p_{r-q-1}} (A_{n+r-1} \cdots p_1 \cdots p_{n+r-2} f_{n+r-1}) x_{r-q}(0).
\end{aligned}$$

4. *Total Population*

The total population at time 't' is given by

$$P(t) = x_1(t) + x_2(t) + \cdots + x_n(t).$$

In consequence of (1.1)b, this equation assumes the form

$$P(t) = x_1(t) + p_1 x_1(t-1) + p_1 p_1 x_1(t-2) + \cdots + p_1 \cdots p_n x_1(t-n). \quad (4.1)$$

If we substitute from (2.3) in this equation, we get $P(t)$ in terms of A_1, ..., A_2. The resulting equation holds when $t>n$. Substituting from (2.6) in (4.1), we get $P(t)$ in terms A_1, A_2 ..., A_{n+r-1} The resulting equation holds when $t>n+r-1$.

When $t=k<n$, (4.1) assumes the form

$$\begin{aligned}
P(t) = {} & x_1(t) + p_1 x_1(t-1) + \cdots + p_1 \cdots p_{k-1} x_1(1) + p_1 \cdots p_k x_1(0) \\
& + p_2 \cdots p_{k+1} x_2(0) + p_3 \cdots p_{k+o} x_3(0) + \cdots + p_{n-k} \cdots p_{n-1} x_{n-k}(0). \quad (4.2)
\end{aligned}$$

References

Bernardelli, H. (1941) : Jour. Burma Research Society, 21, 41-44; 31, 1-10.

Leslie, P. H. (1945) : Biometrika, 32, 183-212.

Leslie, P. H. (1948) : Biometrika, 35, 213-245.

Lewis, P. G. (1942) : Sankhya, 6 95-96.