Mathematical Forum, Vol. VIII, No. 2, Sept. 1991, pp. 11-19

THE IMAGE SYSTEM FOR STOKES FLOW IN A HALF-SPACE WITH SLIP AT THE WALL

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ABSTRACT

The image system for four singularities viz., source, potential-doublet, rotlet and Stokeslet has been obtained. The image system is arrived at through the help of double Fourier transform. Expressions for slip-velocity in all the cases have been derived.

Introduction :

In the construction of solutions of physical problems in fluid mechanics, the role of singularities is very important. The flow of an incompressible viscous fluid at small Reynolds number is governed by Stokes equations, whose fundamental solution is Stokeslet. It exhibits the effect of a concentrated point force in a fluid of infinite extent. Rotlet is another important singularity. It embodies the effect of concentrated point torque. Other associated singularities are source, potential-doublet and stresslet. With the help of these singularities many problems of slow motion have been obtained and discussed [4, 5].

In some problems such as the motion of a particle near a boundary surface, the fluid is not of infinite extent. Therefore, Blake [1, 2] has discussed the singularities in a semi-infinite fluid bounded by a no-slip plane. The case of Stokeslet in a two-fluid space has been discussed by Aderogba [3].

The slip condition where the tangential velocity relative to the solid boundary is proportional to the corresponding viscous stress is another interesting boundary condition. Such conditions have been discussed by Tsien [7] and Schenell [8].

In this paper we will consider four singularities—(i) source (ii) potential-doublet (iii) rotlet and (iv) Stokeslet. Out of these the case of the source has been worked out in detail here, while the case of Stokeslet has been studied earlier by Datta [6]. In order to obtain the

image system we shall express the velocity and pressure in each case as follows :

(1) $\bar{v}=\bar{u}_{SI}+\bar{u}'_{SI}+\bar{w}$ and

(2) $p=p_{SI}+p'_{SI}+q$

where $\bar{u}_{SI}$ and $\bar{u}'_{SI}$ denote the velocity fields due to original and opposite image singularities respectively and p_{SI} and p'_{SI} denote the corresponding pressures. $\bar{w}$ and q are additional parts which we have to calculate. The various singularities forming the image system can be recognized as follows [5] :

(3) $\bar{u}_{SO}(\bar{r})=\text{Source}=\dfrac{\bar{r}}{r^3}j$

(4) $\bar{u}_D(\bar{r}_j\cdot\bar{\alpha})=\text{Potential-doublet}=-\dfrac{\bar{\alpha}}{r^3}+\dfrac{3(\bar{\alpha}\cdot\bar{r})\bar{r}}{r^5}$;

(5) $\bar{u}_{SS}(\bar{r}\,;\,\bar{\alpha},\bar{\beta})=\text{Stresslet}=\left[-\dfrac{\bar{\alpha}\cdot\bar{\beta}}{r^3}+\dfrac{3(\bar{\alpha}\cdot\bar{r})(\bar{\beta}\cdot\bar{r})}{r^5}\right]\bar{r}$;

(6) $\bar{u}_R(\bar{r}\,;\,\bar{\alpha})=\text{Rotlet}=\dfrac{\bar{\alpha}\times\bar{r}}{r^3}$;

(7) $\bar{u}_S(\bar{r}\,;\,\bar{\alpha})=\text{Stokeslet}=\dfrac{\bar{\alpha}}{r}+\dfrac{(\bar{\alpha}\cdot\bar{r})\bar{r}}{r^3}$;

(8) $P_S(\bar{r}\,;\,\bar{\alpha})=2\mu\dfrac{\bar{\alpha}\cdot\bar{r}}{r^3}$

2. *Basic equations* :

The flow at small Reynolds number is governed by Stokes equations

(9) $\mu\nabla^2 V_k=\dfrac{\partial p}{\partial x_k}$;

(10) $\dfrac{\partial v_k}{\partial x_k}=0$; $(k=1, 2, 3)$; $(x_1, x_2, x_3)\equiv(x, y, z)$

where v_k are velocity components, p is the pressure and μ is the coefficient of viscosity.

Let us suppose that the boundary surface at which slipping takes place occupies the plane $z=0$ and the fluid occupies the space $z>0$. Let the singularity be placed at $(0, 0, h)$. The boundary conditions to be satisfied are

(11) $v_1=\sigma\dfrac{\partial v_1}{\partial z}$, $v_2=\sigma\dfrac{\partial v_2}{\partial z}$, $v_3=0$ at $z=0$

where the last condition ensures that both $\frac{\partial v_3}{\partial x}$ and $\frac{\partial v_3}{\partial y}$ vanish on the boundary surface. Also,

(12) $v_1, v_2, v_3 \to 0$ at infinity.

Here σ is the slip coefficient.

3. *Source*:

Let us take the source to be of unit strength. Then the velocity field generated by it may be expressed as follows :

(13) $\bar{u}_{SO} = \frac{\bar{r}}{r^3}$, where $\bar{r} = x\bar{e}_x + y\bar{e}_y + (z-h)\bar{e}_z$

Also, the velocity field due to opposite image source at $(0, 0, -h)$ is

(14) $\bar{u}'_{SO} = \frac{\bar{r}'}{r'^3}$, where $\bar{r}' = x\bar{e}_x + y\bar{e}_y + (z+h)\bar{e}_z$

Now, let us write

(15) $\bar{v} = \bar{u}_{SO} + \bar{u}'_{SO} + \bar{w}$; $p = p_{SO} + p'_{SO} + q$

Therefore, it follows from (9), (10), (11) and (12) that $\bar{w}$ satisfies the equations :

(16) $\mu \nabla^2 w_k \frac{\partial q}{\partial x_k}$;

(17) $\frac{\partial w_k}{\partial x_k} = 0$

together with the boundary conditions

(18) $w_1 - \sigma \frac{\partial w_1}{\partial z} = -\frac{2x}{r_0^3}$, $r_0^2 = x^2 + y^2 + h^2$,

(19) $w_2 - \sigma \frac{\partial w_2}{\partial z} = -\frac{2y}{r_0^3}$;

(20) $w_3 = 0$
at $z = 0$ and

(21) $w_1, w_2, w_3 \to 0$ at infinity

Transforming equations (16) and (17) through the use of double Fourier transform

$$\hat{f}(\alpha_1, \alpha_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\alpha_1 x + \alpha_2 y)} f(x, y)\, dx dy$$

and calculating the constants with the help of transformed boundary conditions (18), (19), (20) and (21), we finally obtain—

$$(22)\quad \hat{w}_1 = -\frac{2i\alpha_1}{\alpha(1+2\sigma\alpha)}\, e^{-\alpha(z+h)} + \frac{2i\alpha_1}{1+2\sigma\alpha}\, z e^{-\alpha(z+h)}\,;$$

$$(23)\quad \hat{w}_2 = -\frac{2i\alpha_2}{\alpha(1+2\sigma\alpha)}\, e^{-\alpha(z+h)} + \frac{2i\alpha_2}{1+2\sigma\alpha}\, z e^{-\alpha(z+h)}\,;$$

$$(24)\quad \hat{w}_3 = \frac{2\alpha}{1+2\sigma\alpha}\, z e^{-\alpha(z+h)}\,;$$

$$(25)\quad \hat{q} = \frac{4\mu\alpha}{1+2\sigma\alpha}\, e^{-\alpha(z+h)}$$

3.1 *Approximate Results*

The inversion of (22)-(25) now leads to the required values ; but since these do not appear in closed form, it has been found convenient to obtain approximate results.

Case I

When σ is small Watson's lemma enables us to make the following approximation :

$$(26)\quad \hat{\bar{w}} = \hat{w}_1\bar{e}_x + \hat{w}_2\bar{e}_y + \hat{w}_3\,\bar{e}_z \simeq \hat{\bar{w}}_0 + \sigma\,\hat{\bar{w}}_\sigma\,;$$

$$(27)\quad \hat{q} \simeq \hat{q}_0 + \sigma\,\hat{q}_\sigma$$

where

$$(28)\quad \hat{\bar{w}}_0 = \frac{2i\alpha_1(\alpha z-1)}{\alpha}\, e^{-\alpha(z+h)}.\,\bar{e}x + \frac{2i\alpha_2(\alpha z-1)}{\alpha} e^{-\alpha(z+h)}\,\bar{e}y + 2\alpha z e^{-\alpha(z+h)}\,\bar{e}z\,;$$

$$(29)\quad \hat{\bar{w}}_\sigma = 4i\alpha_1(1-\alpha z)e^{-\alpha(z+h)}\,\bar{e}x + 4i\alpha_2(1-\alpha z)e^{-\alpha(z+h)}.\,\bar{e}y - 4\alpha^2 z e^{-\alpha(z+h)}\,\bar{e}z\,;$$

$$(30)\quad \hat{q}_0 = 4\mu\alpha e^{-\alpha(z+h)}\,;$$

$$(31)\quad \hat{q}_\sigma = -8\mu\alpha^2 e^{-\alpha(z+h)}$$

Finally applying inversion theorem we get

$$(32)\quad \bar{w}_0 = 2\bar{u}_{SS}(\bar{r}\,;\,\bar{e}_z,\,\bar{e}_z) - 2h\bar{u}_D(\bar{r}\,;\,\bar{e}_z)\,;$$

$$(33)\quad \bar{w}_\sigma = 4\frac{\partial}{\partial z}\,\bar{u}_{SS}(\bar{r}\,;\,\bar{e}_z,\,\bar{e}_z) - 4h\frac{\partial}{\partial z}\,\bar{u}_D(\bar{r}\,;\,\bar{e}_z) - 4\bar{u}_D(\bar{r}\,;\,\bar{e}_z)\,;$$

$$(34)\quad q_0 = -2\frac{\partial}{\partial z}\,P_S(\bar{r}\,;\,\bar{e}_z)\,;$$

$$(35)\quad q_\sigma = -4\frac{\partial^2}{\partial z^2}\,P_S(\bar{r}\,;\,\bar{e}_z)$$

It is seen that after minor changes in notation $(\bar{w}_0, q_0)$ correspond to the no slip case discussed by Blake [2].

Case II

On the other hand if $\sigma \to \infty$, we find that $w_1, w_2, w_3 \to 0$ and so $\bar{w} \to 0$

4. *Potential-doublet*

The problem, in this section has been divided into two parts.

Case (i) *Potential-doublet Parallel to the Surface*

In this case we take

$$\bar{v} = \bar{u}_D(\bar{r}, \bar{e}_x) + \bar{u}_D(\bar{r}', \bar{e}_x) + \bar{w}$$

where $\bar{r}$ and $\bar{r}'$ are as mentioned in (13) and (14). Then the boundary conditions to be satisfied at $z=0$, assume the form

$$(36) \quad w_1 - \sigma \frac{\partial w_1}{\partial z} = \frac{2}{r_0^3} - \frac{6x^2}{r_0^5}\ ;\ w_2 - \sigma \frac{\partial w_2}{\partial z} = -\frac{6xy}{r_0^5}\ ;\ w_3 = 0$$

As in the previous section, an application of double Fourier transform leads to the following approximations, when σ is small.

$$(37) \quad \bar{w}_0 = 2h \frac{\partial}{\partial z} \bar{u}_D(\bar{r}\ ;\ \bar{e}_x) + 2 \frac{\partial}{\partial z} \bar{u}_R(\bar{r}\ ;\ \bar{e}_y) - 2 \frac{\partial}{\partial z} \bar{u}_{SS}(\bar{r}\ ;\ \bar{e}_x, \bar{e}_z)\ ;$$

$$(38) \quad \bar{w}_\sigma = 4 \frac{\partial}{\partial z} \bar{u}_D(\bar{r}\ ;\ \bar{e}_x) + 4h \frac{\partial^2}{\partial z^2} \bar{u}_D(\bar{r}\ ;\ \bar{e}_x)$$
$$+ 4 \frac{\partial^2}{\partial z^2} \bar{u}_R(\bar{r}\ ;\ \bar{e}_y) - 4 \frac{\partial^2}{\partial z^2} \bar{u}_{SS}(\bar{r}\ ;\ \bar{e}_x, \bar{e}_z)\ ;$$

$$(39) \quad q_0 = 2 \frac{\partial^2}{\partial z^2} P_S(\bar{r}\ ;\ \bar{e}_x)\ ;\ q_\sigma = 4 \frac{\partial^3}{\partial z^3} P_S(\bar{r}\ ;\ \bar{e}_x)$$

It is seen that after minor changes in notation $(\bar{w}_0, q_0)$ agrees with the no slip case discussed by Blake [2].

On the other hand if $\sigma \to \infty$, we find that $w_1, w_2, w_3 \to 0$ and so $\bar{w} \to 0$.

Case (ii) *Potential-doublet Perpendicular to the boundary*

In this case we take,

$$\bar{v} = \bar{u}_D(\bar{r}, \bar{e}_z) + \bar{u}_D(\bar{r}'\ ;\ -\bar{e}_z) + \bar{w}$$

Then boundary conditions to be satisfied at $z=0$, are

$$(40) \quad w_1 - \sigma \frac{\partial w_1}{\partial z} = \frac{6hx}{r_0^5}\ ;\ w_2 - \sigma \frac{\partial w_2}{\partial z} = \frac{6hy}{r_0^5}\ ;\ w_3 = 0$$

As in the previous case, when σ is small we have here

(41) $\bar{w}_0 = -2\bar{u}_D(\bar{r}'; \bar{e}_z) + 2\frac{\partial}{\partial z}\bar{u}_{SS}(\bar{r}'; \bar{e}_z, \bar{e}_z) - 2h\frac{\partial}{\partial z}\bar{u}_D(\bar{r}'; \bar{e}_z)$;

(42) $\bar{w}_\sigma = -8\frac{\partial}{\partial z}\bar{u}_D(\bar{r}'; \bar{e}_z) - 4h\frac{\partial^2}{\partial z^2}\bar{u}_D(\bar{r}'; \bar{e}_z) + 4\frac{\partial^2}{\partial z^2}\bar{u}_{SS}(\bar{r}'; \bar{e}_z, \bar{e}_z)$;

(43) $q_0 = -2\frac{\partial^2}{\partial z^2}P_S(\bar{r}'; \bar{e}_z)$; $q_\sigma = -4\frac{\partial^3}{\partial z^3}P_S(\bar{r}'; \bar{e}_z)$

Here again $(\bar{w}_0, q_0)$ agrees with no slip case discussed by Blake [2].

On the other hand if $\sigma \to \infty$, we see that $w_1, w_2, w_3 \to 0$ and so $\bar{w} \to 0$.

5. *Rotlet*

Case (i) *Rotlet parallel to the surface*

Here we write,

$$\bar{v} = \bar{u}_R(\bar{r}, \bar{e}_x) + u_R(\bar{r}', -\bar{e}_x) = \bar{w}$$

Then the boundary conditions to be satisfied at $z=0$ are

(44) $w_1 - \sigma\frac{\partial w_1}{\partial z} = 0$; $w_2 - \sigma\frac{\partial w_2}{\partial z} = -\frac{2h}{r_0^3}$; $w_3 = 0$

As in the previous case, when σ is small we have here,

(45) $\bar{w}_0 = 2h\bar{u}_D(\bar{r}'; \bar{e}_y) - 2\bar{u}_{SS}(\bar{r}'; \bar{e}_y, \bar{e}_z)$;

(46) $\bar{w}_\sigma = 4\bar{u}_D(\bar{r}'; \bar{e}_y) + 4h\frac{\partial}{\sigma z}\bar{u}_D(\bar{r}'; \bar{e}_y)$

$$-4\frac{\partial}{\partial z}\bar{u}_{SS}(\bar{r}'; \bar{e}_y, \bar{e}_z) - 2\frac{\partial}{\partial x}\bar{u}_R(\bar{r}; \bar{e}_z);$$

(47) $q_0 = 2\frac{\partial}{\partial z}P_S(\bar{r}'; \bar{e}_y)$; $q_\sigma = 4\frac{\partial^2}{\partial z^2}P_S(\bar{r}'; \bar{e}_y)$

It is seen that $(\bar{w}_0; q_0)$ agrees with no slip case discussed by Blake [2].

Again, if $\sigma \to \infty$, we find that $w_1, w_2, w_3 \to 0$ and so $\bar{w} \to 0$.

Case (ii) *Rotlet Perpendicular to the boundary*

Here we have

$$\bar{v} = \bar{u}_R(\bar{r}, \bar{e}_z) + \bar{u}_R(\bar{r}', -\bar{e}_z) + \bar{w}$$

Then the boundary conditions to be satisfied at $z=0$ are,

(48) $w_1 - \sigma\frac{\partial w_1}{\partial z} = -\frac{6\sigma hy}{r_0^5}$; $w_2 - \sigma\frac{\partial w_2}{\partial z} = \frac{6\sigma hx}{r_0^5}$; $w_3 = 0$

As in the previous case, when σ is small we get

(49) $\bar{w} = -2\sigma \frac{\partial}{\partial z} \bar{u}_R(\bar{r}' ; \bar{e}_z) ; q=0$

Here again $(\bar{w}_0, q_0)$ agrees with no-slip case discussed by Blake [2].

On the other hand when $\sigma \to \infty$ we find that

(50) $\bar{w} \simeq \cdot 2 \frac{x\bar{e}_y - y\bar{e}_x}{r'^3}$

6. *Stokeslet*

Stokeslet has been discussed by Datta [6] and here we briefly add his results for the sake of completion.

6.1 *Stokeslet parallel to the surface*

When σ is small, we have here

(51) $\bar{w}_0 = -2h^2 \bar{u}_D(\bar{r}' ; \bar{e}_x) + 2h\bar{u}_{SS}(\bar{r}' ; \bar{e}_x, \bar{e}_z) - 2h\bar{u}_R(\bar{r}' ; \bar{e}_y) ;$

(52) $\bar{w}_\sigma = 4\bar{u}_{SS}(\bar{r}' ; \bar{e}_x, \bar{e}_y) + 4h\left[\frac{\partial}{\partial z} \bar{u}_{SS}(\bar{r}' ; \bar{e}_x, \bar{e}_y) - \frac{\partial}{\partial z} \bar{u}_R(\bar{r}' ; \bar{e}_y) - h\frac{\partial}{\partial z} \bar{u}_D(\bar{r}' ; \bar{e}_x) - \bar{u}_D(\bar{r}' ; \bar{e}_x)\right] ;$

(53) $q_0 = -2h \frac{\partial}{\partial x} P_S(\bar{r}' ; \bar{e}_z) ;$

(54) $q_\sigma = -4h \frac{\partial}{\partial x}\left(1 - h\frac{\partial}{\partial z}\right) P_S(\bar{r}' ; \bar{e}_z)$

On the other hand when $\sigma \to \infty$, we have

(55) $\bar{w} \simeq 2\left[\frac{\bar{e}_x}{r'} + \frac{x\bar{r}'}{r'^3}\right]$

6.2 *Stokeslet perpendicular to the boundary*

When σ is small we have,

(56) $\bar{w}_0 = 2h^2 \bar{u}_D(\bar{r}' ; \bar{e}_z) - 2h\bar{u}_{SS}(\bar{r}' ; \bar{e}_z, \bar{e}_z) ;$

(57) $\bar{w}_\sigma = 4h\bar{u}_D(\bar{r}' ; \bar{e}_z) + 4h^2 \frac{\partial}{\partial z} \bar{u}_D(\bar{r}' ; \bar{e}_z) - 4h \frac{\partial}{\partial z} \bar{u}_{SS}(\bar{r}' ; \bar{e}_z, \bar{e}_z)$

Also when $\sigma \to \infty$, we see that $w_1, w_2, w_3 \to 0$ and so $\bar{w} \to 0$.

7. *Slip-velocity*

The slip-velocity at the wall $z=0$ is an interesting parameter. In the case of source the slip-velocity is given by

(58) $\bar{V}_S = \sigma \bar{w}_\sigma \,|_{z=0} = \frac{12\sigma h}{r_0^5} (x\bar{e}_x + y\bar{e}_y)$

It is interesting to note that $\bar{V}_S$ increases first with the increase of h and then it decreases, ultimately vanishing as $h \to \infty$. For maximum slip effect to occur at the point (x, y) the source has to be placed at the height $\sqrt{x^2+y^2}$.

Since small σ expansion may not be tenable when h is small, the initial increase at small h must be taken with reservation. Thus to conclude we note that as the source is moved away from the wall the general effect of the slip decreases.

Similarly we can discuss the slip-velocities in other cases also. The slip-velocities parallel and perpendicular to the wall are denoted by V_{S_1} and V_{S_2} and they are as follows :

In the case of potential-doublet we have,

$$(59) \quad \bar{V}_{S_1} = \frac{60\sigma h x}{r_0^7}(x\bar{e}_x + y\bar{e}_y) - \frac{12\sigma h}{r_0^5}\bar{e}_x ;$$

$$(60) \quad \bar{V}_{S_2} = \left[\frac{12\sigma}{r_0^5} - \frac{60\sigma h^2}{r_0^7}\right](x\bar{e}_x + y\bar{e}_y)$$

In the case of rotlet we have,

$$(61) \quad \bar{V}_{S_1} = -\frac{6\sigma\bar{e}_y}{r_0^3} + \frac{6\sigma x}{r_0^5}(x\bar{e}_y - y\bar{e}_x) + \frac{12\sigma h^2\bar{e}_y}{r_0^5} ;$$

$$(62) \quad \bar{V}_{S_2} = \frac{6\sigma h}{r_0^5}(x\bar{e}_y - y\bar{e}_x)$$

Lastly in the case of Stokeslet we have,

$$(63) \quad \bar{V}_{S_1} = \frac{12\sigma h x}{r_0^5}(x\bar{e}_x + y\bar{e}_y)$$

$$(64) \quad \bar{V}_{S_2} = -\frac{12\sigma h^2}{r_0^5}(x\bar{e}_x + y\bar{e}_y)$$

Acknowledgement

The author is indebted to Dr. Sunil Datta, for his valuable guidance in the preparation of this paper.

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