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FLOW OF A SECOND-ORDER FLUID THROUGH A CIRCULAR PIPE AND ITS SURROUNDING POROUS MEDIUM

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ABSTRACT

The flow of a second-order fluid through a circular pipe and its surrounding porous medium has been discussed when (i) the surrounding region extends to a large distance and (ii) it is bounded by an impervious co-axial circular cylinder. The flow equation in the pipe has been solved for a second-order fluid and in the porous medium the solution has been found by using modified Brinkman [1] equation. The solution at the interface has been matched by considering the fact that the velocity and shearing stresses are reduced in the porous medium. It has been found that there exists a constant seepage flow at a large distance from the axis of the pipe in the infinite porous medium. This seepage flow increases with the increase of the permeability of the medium. The non-Newtonian terms increase the force experienced by a particular layer of the porous medium and it also increase the angle that this force makes with the surface of the pipe.

1. *Introduction*

Beavers and Joseph [2] have suggested the boundary condition at the interface by taking the flow governed by Darcy law in the porous medium and by Navier-Stokes equation in the free fluid region. Brinkman [1] has suggested the modification of the Darcy law in which case the boundary condition at the interface suggested by Beavers and Joseph [2] is not sufficient. Verma [3] has solved this type of problem by equating the velocity and the shearing stress at the interface calculated by the solution of Navier-Stokes equation to their corresponding values obtained from the solution of Brinkman equation in the porous medium. This is not justified as velocity and shearing stresses decrease when the flow enters the porous medium. Hence Rudraiah [4] has suggested that in porous medium both velocity and shearing stress must reduce in constant ratios.

The study of flow of a non-Newtonian fluid past a porous medium has been a topic of considerable research interest because in many situations small particles of solids are found to be present in a fluid such as in blood plasma and in crude oil in which case the fluid behaves as a non-Newtonian one. Most common model for such a fluid is that of a second-order fluid whose constitutive equation has been given by Coleman and Noll [5] as :

$$\tau_{ij}+p\delta_{ij}=\mu_1 A_{(1)ij}+\mu_2 A_{(2)ij}+\mu_3 A_{(1)ik}\, A_{(1)jk}, \tag{1}$$

where

$$\left.\begin{array}{l} A_{(1)ij}=v_{i,\,j}+v_{j,\,i} \\ A_{(2)ij}=a_{i,\,j}+a_{j,\,i}+2v^m{}_{,\,i}\, v_{m,\,j} \end{array}\right\}, \tag{2}$$

where τ_{ij} is the stress tensor, v_i, a_i are respectively velocity and acceleration vectors, μ_1, μ_2, μ_3 are material constants, p is the hydrostatic pressure and a comma denotes a co-variant differentiation with respect to the symbol following it.

In this paper we have discussed the flow of a second-order fluid governed by constitutive equations (1) and (2) through a circular pipe and also through its surrounding porous medium taking two different cases. When the porous medium is (i) infinite in extent (ii) of finite thickness. For discussing the flow of a second-order fluid in the porous medium we have generalized the Brinkman [1] equation by taking the stresses defined by (1) and (2). This generalized Brinkman equation together with the boundary conditions suggested by Rudraiah [4] has been used for obtaining the solution of the velocity profile in the porous region. The graph of the velocity profile against the distance from the central line of the pipe has been plotted. The slip-velocity at the interface for different values of porosity against a permeability parameter has been given in table—1. We have derived the expressions for the force experienced by a layer in the porous region and the angle that this force makes with the surface of the pipe. The expression for velocity has also been calculated for the case when an impervious surface co-axial to the pipe is present. The force experienced by the impervious surface together with the angle that this force makes with the surface of the pipe have been given in table—2 for various values of the permeability and non-Newtonian parameters. Our results can be applied to the flow of blood through arteries and veins bounded by flesh which is porous. The seepage of the oil at a large distance can be predicted when the oil flows through cylindrical cavities bounded by porous medium.

2. *Formulation of the Problem*

We consider the flow of a second-order fluid governed by the constitutive equations (1) and (2) through a circular pipe $r=a$ and through its surrounding infinite porous medium $r \geqslant a$. The case when the porous medium is bounded by an impervious co-axial cylinder $r=b$, $(b>a)$ is also taken into account. We assume the velocity components u, v in the direction of r, θ respectively to be zero and the only non-vanishing velocity component w in the direction of z to be a function of r only, i.e.,

$$u=0, \qquad v=0, \qquad w=w(r) \tag{3}$$

Inside the circular pipe the equation of motion and the equation of continuity are given by

$$\rho\, v_j\, v_{i,\,j}=\tau_{ij,\,j} \tag{4}$$

$$v_{i,\,i}=0 \tag{5}$$

where ρ denotes the fluid density. The form of velocity components (3) satisfies equation (5) identically. Substituting equation (1), (2) and (3) in equation (4), we get

$$\frac{\partial p}{\partial r}=(2\mu_2+\mu_3)\,\frac{dw}{dr}\left(2\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) \tag{6}$$

$$\frac{\partial p}{\partial \theta}=0 \tag{7}$$

$$\frac{\partial p}{\partial z}=\mu_1\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) \tag{8}$$

Outside the circular pipe $r\geqslant a$, the equation of motion (4) is modified by adding Brinkman [1] term as

$$\rho v_j\, v_{i,\,j}=\tau_{ij,\,j}-\frac{\mu_1}{k}\,v_i \tag{9}$$

where k is the permeability of the porous medium. The equation of continuity remains unchanged. Taking

$$\bar{u}=0,\ \bar{v}=0,\ \bar{w}=\bar{w}(r) \tag{10}$$

as velocity components in the porous medium and substituting equation (1), (2) and (10) in equation (9) we get

$$\frac{\partial \bar{p}}{\partial r}=(2\mu_2+\mu_3)\,\frac{d\bar{w}}{dr}\left(2\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right) \tag{11}$$

$$\frac{\partial \bar{p}}{\partial \theta}=0 \tag{12}$$

$$\frac{\partial \bar{p}}{\partial z}=\mu_1\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)-\frac{\mu_1}{k}\,\bar{w} \tag{13}$$

where $\bar{p}$ is hydrostatic pressure in the porous medium. τ_{rz} and $\bar{\tau}_{rz}$ representing shearing stresses in free fluid region and in porous region respectively are given by

$$\tau_{rz}=\mu_1 \frac{dw}{dr}, \quad \bar{\tau}_{rz}=\mu_1 \frac{d\bar{w}}{dr} \tag{14}$$

The boundary conditions at the interface are given by the fact that the velocity w and shearing stress τ_{rz} are reduced by constant factors ϕ and $\lambda\phi$ respectively, i.e.,

$$\bar{w}=\phi w, \tag{15}$$

$$\frac{d\bar{w}}{dr}=\lambda\phi \frac{dw}{dr} \tag{16}$$

3. *Infinite surrounding medium*

The case when the flow is taken in a circular pipe and the surrounding porous medium which is infinite in extent, the boundary conditions are given by :

$$w \text{ is finite at } r=0 \tag{17}$$

$$\bar{w} \text{ is finite at } r \to \infty \tag{18}$$

Equations (6), (7) and (8) shows that the pressure p is independent of θ and

$$\frac{\partial p}{\partial z}=\text{constant}=\mu_1 A \text{ (say)} \tag{19}$$

Taking this form for the pressure, the equation of motion in the pipe is given by

$$\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}=A \tag{20}$$

The solution of equation (20) is

$$w=\frac{A}{4}r^2+B\log_e r+C \tag{21}$$

Assuming the same pressure gradient $\frac{\partial \bar{p}}{\partial z}=A\mu_1$ the equation of motion in the porous medium becomes

$$\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}+\frac{\bar{w}}{k}=A \tag{22}$$

whose solution is

$$\bar{w}=DI_0\left(\frac{r}{\sqrt{k}}\right)+EK_0\left(\frac{r}{\sqrt{k}}\right)-Ak \tag{23}$$

where I_0 and K_0 are modified Bessels functions of zeroth order. The solution of equation (21) and (23) satisfying boundary conditions (15), (16), (17) and (18) can be written respectively as

$$w'(\eta)=\frac{w}{-\frac{Aa^2}{4}}=(1-\eta^2)+\frac{2\lambda K_0(\sigma)}{\sigma K_1(\sigma)}+\frac{4}{\phi\sigma^2} \tag{24}$$

and

$$\bar{w}'=\frac{\bar{w}}{-\frac{Aa^2}{4}}=\frac{2\lambda\phi K_0(\sigma\eta)}{\sigma K_1(\sigma)}+\frac{4}{\sigma^2} \tag{25}$$

where $\eta=\frac{r}{a}$ and $\sigma=\frac{a}{\sqrt{k}}$

4. *The case when the Porous medium fills the region* $a\leqslant r\leqslant b$

In this case all the boundary conditions (15), (16) and (17) remain the same except (18) which becomes

$$\bar{w}=0 \text{ at } r=b \tag{26}$$

The solution of equations (21) and (23) satisfying boundary conditions (15), (16), (17) and (26) can be written respectively as :

$$w''=\frac{w}{-\frac{Aa^2}{4}}=(1-\eta^2)+\frac{4}{\phi\sigma^2}-MK_0(\sigma)-NI_0(\sigma) \tag{27}$$

and

$$\bar{w}''=\frac{\bar{w}}{-\frac{Aa^2}{4}}=\frac{4}{\sigma^2}-\phi\{MK_0(\sigma\eta)+NI_0(\sigma\eta)\} \tag{28}$$

where

$$M=\frac{2[2I_1(\sigma)-\lambda\phi\sigma I_0(\sigma h)]}{\phi\sigma^2[I_0(\sigma h)K_1(\sigma)+K_0(\sigma h)I_1(\sigma)]}$$

$$N=\frac{2[2K_1(\sigma)+\lambda\phi\sigma K_0(\sigma h)]}{\phi\sigma^2[I_0(\sigma h)K_1(\sigma)+K_0(\sigma h)I_1(\sigma)]}$$

and $h=\frac{b}{a}$

5. *Discussion*

The velocity profile has been drawn in fig. (i) for $\lambda=0{\cdot}077$, $\phi=0{\cdot}39$ and for different values of σ inside and outside the pipe surrounding by infinite porous medium. It is observed that the velocity inside the pipe and outside it decreases with the increase of the permeability parameter σ. Maximum velocity at the axis of the pipe is also enhanced due to the porosity of the boundary. It is also observed that there is a boundary layer

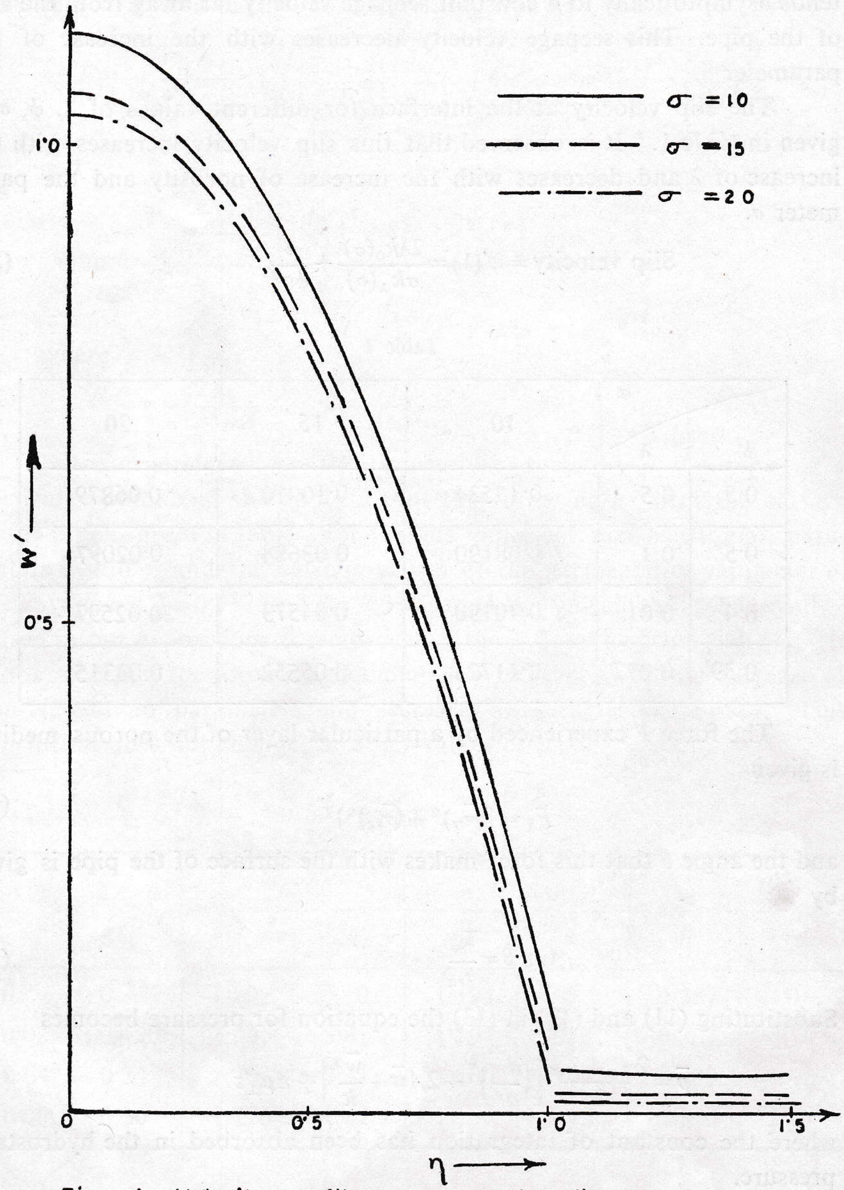

Fig. 1 Velocity profile against the distance from the centre of the pipe

formation near the interface in the porous medium and the velocity tends asymptotically to a constant seepage velocity far away from the axis of the pipe. This seepage velocity decreases with the increase of the parameter σ.

The slip velocity at the interface for different values of λ, ϕ, σ is given in table 1. It is observed that this slip velocity increases with the increase of λ and decreases with the increase of porosity and the parameter σ.

$$\text{Slip velocity} = w'(1) = \frac{2\lambda k_0(\sigma)}{\sigma k_1(\sigma)} + \frac{4}{\phi\sigma^2} \tag{29}$$

Table 1

ϕ	λ \ σ	10	15	20
0·5	0·5	0·17534	0·10010	0·06879
0·5	0·1	0·08190	0·03684	0·02097
0·4	0·01	0·10190	0·04573	0·02597
0·39	0·077	0·11725	0·05552	0·03315

The force F experienced by a particular layer of the porous medium is given

$$F_1 = \{(\bar{\tau}_{rr})^2 + (\bar{\tau}_{rz})^2\}^{\frac{1}{2}} \tag{30}$$

and the angle θ that this force makes with the surface of the pipe is given by

$$\tan\theta = \frac{\bar{\tau}_{rr}}{\bar{\tau}_{rz}} \tag{31}$$

Substituting (11) and (12) in (13) the equation for pressure becomes

$$\bar{p} = \frac{2\mu_2 + \mu_3}{2}\left[\left(\frac{d\bar{w}}{dr}\right)^2 + 2A\bar{w} + \frac{\bar{w}^2}{k}\right] + A\mu_1 z \tag{32}$$

where the constant of integration has been absorbed in the hydrostatic pressure.

The expression for normal stress $\bar{\tau}_{rr}$ in the porous region now takes the form

$$\bar{\tau}_{rr} = \frac{2\mu_2 + \mu_3}{2}\left[\left(\frac{d\bar{w}}{dr}\right)^2 - 2A\bar{w} - \frac{\bar{w}^2}{k}\right] - A\mu_1 z$$

Substituting (14) and (33) in (30) and (31) the expression for F and $\tan\theta$ at $z=0$ in non-dimensional form are given as

$$F=\frac{F_1}{\frac{A a \mu_1}{4}}=\sqrt{\left[\frac{2\alpha+\beta}{2}\left\{\left(\frac{d\bar{w}'}{d\eta}\right)^2+8\bar{w}'-\sigma^2\bar{w}'^2\right\}\right]^2+\left(\frac{d\bar{w}'}{d\eta}\right)^2}$$

and

$$\tan\theta=\frac{-\frac{2\alpha+\beta}{2}\left[\left(\frac{d\bar{w}'}{d\eta}\right)^2+8\bar{w}'-\sigma^2\bar{w}'^2\right]}{\frac{d\bar{w}'}{d\eta}}$$

where

$$\alpha=\frac{32\mu_2}{A^2a^2\mu_1^2}\quad,\quad\beta=\frac{32\mu_3}{A^2a^2\mu_1^2}$$

Force experienced by the impervious surface of the pipe at a particular point and the angle θ that this force makes with the surface of the pipe is given in table 2 for various values of non-Newtonian parameters α and β and for various values of the permeability parameter σ. We have taken $\lambda=0{\cdot}077$ and $\phi=0{\cdot}39$ as used by Rudraiah [4]. The symbol λ used in our expressions is reciprocal of the λ used by Rudraiah [4]. It is observed that the force F and the angle θ increases with the increase of non-Newtonian parameters and decreases with the increase of σ. This conclusion does not change for other values of z.

Table 2

		$\sigma=2$		$\sigma=3$		$\sigma=10$	
α	β	F	θ	F	θ	F	θ
0	0	1·692	0	1·194	0	0·387	0
−0·02	0·120	1·696	3·87	1·195	2·73	0·389	0·89
−0·04	0·24	1·707	7·71	1·200	5·45	0·389	1·78
+0·06	0·36	1·726	11·47	1·206	8·15	0·390	2·67

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