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AXIALLY SYMMETRICAL JET MIXING OF A COMPRESSIBLE DUSTY FLUID

N. Dutta and S. K. Das
Department of Mathematics
Indian Institute of Technology
Kharagpur-721302 (India)

ABSTRACT

Compressible laminar jet mixing of a dusty fluid issuing from a circular jet has been considered by using Saffman's model for dusty fluid. The carrier fluid has been considered to be compressible and the flow of the jet has been assumed to be under full expansion from a nozzle. The governing equations have been linearized by the method of small perturbation and have been solved by using successively Hankel and Laplace transform technique. A numerical quadrature method has been employed to calculate the integrals giving the velocity, temperature and density of the fluid and particle phase respectively. The magnitudes of velocity and temperature of the fluid phase are greater than those of the particle phase near the exit of the nozzle whereas the reverse results occur in the downstream direction. The increase of the concentration parameter α increases the axial velocity and temperature for fluid as well as particle phase near the jet axis. Far away from the nozzle exit, the particle velocity, temperature and density decrease and attain the value of the surrounding stream.

1. *Introduction* :

The mixing of an axisymmetric jet of a compressible dusty fluid exhausting into a uniform stream requires much attention in view of its application in atmospheric pollution, rocket motor exhaust, etc. Pai [1] has studied jet mixing of a compressible fluid issuing from a finite opening. The flow of the jet is assumed to be under full expansion from a nozzle, i.e., the pressure of the flow at the exit of the nozzle is exactly equal to that of the surrounding stream.

To simplify the inherent complexity of the dynamics of the dusty fluid, certain assumptions are made in most of the available literature (cf. [2]—[4]). Soo [5] considered circular jet mixing of a dusty fluid with clear fluid where the carrier fluid is considered to be incompressible.

He assumed the dusty fluid to be of highly dilute suspension and solved the problem by dropping the drag force term in fluid momentum equation and by neglecting particle momentum equation in the transverse direction. Datta and Das [6] have solved the problem of flow and heat transfer of axisymmetric jet mixing of an incompressible dusty fluid considering the drag force and the momentum equation in the transverse direction of the particle phase.

The present study considers the carrier fluid to be compressible and the governing equations have been linearized assuming the flow to be laminar. The basic equations have been solved by perturbation method and by using double transform technique of applying successively Hankel and Laplace transform. A numerical method has been used to compute the integrals representing the velocity, temperature and density in the mixing region of the jet.

2. *Basic Equations* :

Since in the present case, the flow is of boundary layer type the equations governing the steady two-phase axially symmetrical compressible flow can be written as

$$\frac{\partial}{\partial z}(\rho r u)+\frac{\partial}{\partial r}(\rho r v)=0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right)+\rho_p\frac{(u_p-u)}{\tau_m}, \tag{2}$$

$$\rho\left[u\frac{\partial(c_pT)}{\partial z}+v\frac{\partial(c_pT)}{\partial r}\right]=\frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial T}{\partial r}\right)+\mu\left(\frac{\partial u}{\partial r}\right)^2+\rho_p\left(\frac{u_p-u}{\tau_m}\right)^2+\frac{\rho_p c_s(T_p-T)}{\tau_T} \tag{3}$$

$$p=\rho RT, \tag{4}$$

$$\frac{\partial}{\partial z}(r\rho_p u_p)+\frac{\partial}{\partial r}(r\rho_p v_p)=0, \tag{5}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial z}+v_p\frac{\partial u_p}{\partial r}\right)=-\rho_p\frac{(u_p-u)}{\tau_m}, \tag{6}$$

$$\rho_p\left(u_p\frac{\partial v_p}{\partial z}+v_p\frac{\partial v_p}{\partial r}\right)=-\left(\frac{v_p-v}{\tau_m}\right), \tag{7}$$

$$\rho_p\left[u_p\frac{\partial}{\partial z}(T_pc_s)+v_p\frac{\partial(T_pc_s)}{\partial r}\right]=-\frac{\rho_pc_s(T_p-T)}{\tau_T} \tag{8}$$

where z and r are co-ordinates along and perpendicular to the jet axi respectively and the origin has been taken at the nozzle exit which is th centre of the nozzle. (u, v) and (u_p, v_p) are the velocity components o the fluid and particle phase respectively, T and T_p are the fluid and particle temperature respectively, ρ and ρ_p are density of the fluid and particle phase respectively, c_p is the specific heat at constant pressure for the gas and c_s is the specific heat of the solid particles, τ_m and τ_T are the mementum and thermal equilibration time respectively, μ is the coefficient of viscosity for the fluid phase. The equations (1)–(8) have been written using Saffman's [2] model and are valid for spherical dust particles of a few microns radius. The equation of state (Eqn. 4) arises due to the compressible nature of carrier fluid.

Assuming the flow from the nozzle to be under full expansion, no pressure variation occurs throughout the flow and the velocity and temperature of the jet differ only slightly from that of the surrounding stream. Then it is possible to write

$$u=u_o+u_l,\ v=v_l,\ u_p=u_{po}+u_{pl},\ v_p=v_{pl},\ T=T_o+T_l,$$
$$T_p=T_{po}+T_{pl} \text{ and } \rho_p=\rho_{po}+\rho_{pl} \text{ with } u_l << u_o,$$
$$u_{pl} << u_{po},\ T_l << T_o,\ T_{pl} << T_{po},\ \rho_{pl} << \rho_{po}, \tag{9}$$

where u_o, u_{po}, T_o, T_{po}

are uniform velocity and temperature at the nozzle exit.

Introducing the following nondimensional quantities,

$$\bar{z}=z/\lambda,\ \bar{r}=\frac{r}{\sqrt{\tau_m v_o}},\quad \bar{u}=\frac{u}{u_o},\quad \bar{v}=v\sqrt{\frac{\tau_m}{v_o}},\quad \bar{u}_p=\frac{u_p}{u_o},$$
$$\bar{v}_p=v_p\sqrt{\frac{\tau_m}{v_o}},\quad \bar{T}=\frac{T}{T_o},\quad \bar{T}_p=\frac{T_p}{T_o},\quad \bar{\rho}_p=\frac{\rho_p}{\rho_{po}},\quad \alpha=\frac{\rho_{po}}{\rho}=\text{const.}$$
$$\lambda=\tau_m u_o,\ \tau_m=\tfrac{2}{3}\frac{c_p}{c_s}\frac{l}{Pr}\tau_T,\ Pr=\frac{\mu c_p}{K} \tag{10}$$

and under the above assumptions the equations (1)–(8) can be written in the nondimensional linearized form after dropping the bars and suffix l, as

$$u_o\frac{\partial T}{\partial z}=T_o\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)+\frac{vT_o}{r}, \tag{11}$$

$$u_o\frac{\partial u}{\partial z}=\frac{T_o^{3/2}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\alpha\rho_{po}(u_p-u), \tag{12}$$

$$u_o\frac{\partial T}{\partial z}=\frac{T_o^{3/2}}{Pr}\frac{l}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\tfrac{2}{3}\frac{\alpha}{Pr}\rho_{po}(T_p-T), \tag{13}$$

$$r\left(\rho_{po}\frac{\partial u_p}{\partial z}+u_{po}\frac{\partial \rho_p}{\partial z}\right)+\rho_{po}\frac{\partial}{\partial r}(rv_p)=0, \tag{14}$$

$$u_{po}\frac{\partial u_p}{\partial z}=-(u_p-u), \tag{15}$$

$$u_{po}\frac{\partial v_p}{\partial z}=-(v_p-v), \tag{16}$$

$$u_{po}\frac{\partial T_p}{\partial z}=-\tfrac{2}{3}\,\frac{c_p}{c_s}\frac{1}{Pr}(T_p-T), \tag{17}$$

The equation (11) has been obtained by the substitution of equation of state to the equation of continuity. Since the pressure is assumed to be constant and the carrier fluid is considered to be compressible, ρ, μ, and K cannot be regarded as constants but vary with gas temperature T.

Therefore, we write,

$$\frac{\rho}{\rho_o}=\frac{T_o}{T}=\frac{l}{T},\quad \frac{\mu}{\mu_o}\sqrt{\frac{T}{T_o}}=\sqrt{T} \tag{18}$$

The problem now is to solve the equations (11)–(17) under the specified nozzle exit and boundary conditions given below,

$$u(0,r)=\begin{cases} u_{10}, & r\leqslant 1\\ 0, & r>1\end{cases} \tag{19}$$

$$\frac{\partial u}{\partial r}(z,0)=0,\quad u(z,\infty)=0 \tag{20}$$

$$v(0,r)=0 \tag{21}$$

$$u_p(0,r)=\begin{cases} u_{p10}, & r\leqslant 1\\ 0, & r>1\end{cases} \tag{22}$$

$$v_p(0,r)=0 \tag{23}$$

$$T(0,r)=\begin{cases} T_{10}, & r\leqslant 1\\ 0, & r>1\end{cases} \tag{24}$$

$$\frac{\partial T}{\partial r}(z,0)=0,\quad T(z,\infty)=0 \tag{25}$$

$$T_p(0,r)=\begin{cases} T_{p10}, & r\leqslant 1\\ 0, & r>1\end{cases} \tag{26}$$

$$\rho_p(0,r)=\begin{cases} \rho_{p10}, & r\leqslant 1\\ 0, & r>1\end{cases} \tag{27}$$

3. *Method of Solution* :

The equations (11)–(17) are highly coupled partial differential equations in the seven unknowns u, u_p, v, v_p, T, T_p and ρ_p. We use a double transform technique of first employing Hankel transform with respect to the variable r and then Laplace transform with respect to variable z.

Since the detail results of this transforms are very complicated we show the method of solution for two equations (12) and (15) for the unknowns u and u_p.

Taking Hankel transform of both sides, the equations (12) and (15) become

$$\frac{d\bar{u}}{dz}+A\bar{u}=B\bar{u}_p \tag{28}$$

$$\frac{d\bar{u}_p}{dz}+C\bar{u}_p=C\bar{u} \tag{29}$$

where $A=\dfrac{T_o^{3/2}\,\xi+\alpha\,\rho_{po}}{u_o}$, $B=\dfrac{\alpha\,\rho_{po}}{u_o}$, $C=\dfrac{l}{u_{po}}$

and $$\bar{u}=\int_0^\infty ru\,J_0\,(\xi r)\,dr,\ \bar{u}_p=\int_0^\infty ru_p\,J_0\,(\xi r)\,dr \tag{30}$$

Using the Laplace transform of both sides with respect to z and solving for $\bar{u}^*$ and $\bar{u}_p^*$ we get

$$\bar{u}^*=\frac{(s+C)\bar{u}(0)+B\bar{u}_p(0)}{s^2+s(A+C)+C(A-S)} \tag{31}$$

$$\bar{u}_p^*=\frac{\bar{u}_p(0)}{(s+C)}+\frac{C\bar{u}(0)}{s^2+s(A+C)+C(A-B)}+\frac{BC\bar{u}_p(0)}{(S+C)[s^2+s(A+C)+C(A-B)]} \tag{32}$$

where $$\bar{u}(0)=\frac{\bar{u}_{10}\,J_1(\xi)}{\xi},\quad \bar{u}_p(0)=\frac{u_{p_{10}}\,J_1(\xi)}{\xi} \tag{33}$$

and $$\bar{u}^*=\int_0^\infty \bar{u}e^{-sz}\,dz,\ \bar{u}_p^*=\int_0^\infty \bar{u}_p\,e^{-sz}\,dz \tag{34}$$

The bars and stars denote the Hankel and Laplace transforms with respect to r and z respectively. The inversion of (31) and (32) gives

$$u=\int_0^\infty\left[u_{10}\cosh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)+\left(\tfrac{1}{2}(C-A)u_{10}+Bu_{p_{10}}\right)\frac{\sinh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)}{\sqrt{BC+\left(\frac{A-C}{2}\right)^2}}\right]e^{\frac{-(A+C)z}{2}}\,J_0(\xi r)J_1(\xi)d\xi \tag{35}$$

$$u_p=\int_0^\infty\left[u_{p_{10}}\cosh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)+\left(Cu_{10}+\tfrac{1}{2}(A-C)u_{p_{10}}\right)\frac{\sinh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)}{\sqrt{BC+\left(\frac{A-C}{2}\right)^2}}\right]e^{\frac{-(A+C)z}{2}}\,J_0\,(\xi r)\,J_1(\xi)\,d\xi \tag{36}$$

Similarly, taking Hankel and Laplace transform of the equations (11), (13), (14), (16), (17) and using the relevant boundary conditions (19)–(27) and solving for T, T_p, v, v_p and ρ_p we get

$$T=\int_0^\infty \left[T_{10} \cosh\left(\sqrt{EF+\frac{(D-F)^2}{2}}\,z\right)+\left(ET_{p_{10}}-\left(\frac{D-F}{2}\right)T_{10}\right) \frac{\sinh\left(\sqrt{EF+\left(\frac{D-F}{2}\right)^2}\,z\right)}{\sqrt{EF+\left(\frac{D-F}{2}\right)^2}}\right] e^{-\frac{(D+F)}{2}z} J_0(\xi)\,J_1(\xi)\,d\xi \tag{37}$$

$$T_p=\int_0^\infty \left[T_{p_{10}} \cosh\left(\sqrt{EF+\left(\frac{D-F}{2}\right)^2}\,z\right)+\left(FT_{10}+\left(\frac{D-F}{2}\right)T_{p_{10}}\right) \frac{\sinh\left(\sqrt{EF+\left(\frac{D-F}{2}\right)^2}\,z\right)}{\sqrt{EF+\left(\frac{D-F}{2}\right)^2}}\right] e^{-\frac{(D+F)}{2}z} J_0(\xi r)\,J_1(\xi)\,d\xi \tag{38}$$

To obtain v, v_p and ρ_p we substitute $v=\frac{\partial f_1}{\partial r}$, $v_p=\frac{\partial f_2}{\partial r}$ in (11), (14) and (16) where f_1 and f_2 are continuous functions of z and r, and take the first order Hankel and Laplace transform successively.

$$v=\int_0^\infty \left[\frac{u_0}{T_0}\left[(ETp_{10}-DT_{10})\cosh\left(\sqrt{EF+\left(\frac{D-F}{2}\right)^2}\,z\right)-\left[F(D-E)T_{10} +\left(\frac{D+F}{2}\right)(ETp_{10}-DT_{10})\right]\frac{\sinh\left(\sqrt{EF+\left(\frac{D-F}{2}\right)^2}\,z\right)}{\sqrt{EF+\left(\frac{D-F}{2}\right)^2}}\right] e^{-\frac{(D+F)}{2}z} \right.$$
$$+\left[\left[C(A-B)u_{10}+\left(\frac{A+C}{2}\right)(Bu_{p_{10}}-Au_{10})\right]\frac{\sinh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)}{\sqrt{BC+\left(\frac{A-C}{2}\right)^2}} -(Bu_{p_{10}}-Au_{10})\cosh\left(\sqrt{BC+\left(\frac{A-C}{2}\right)^2}\,z\right)\right] e^{-\left(\frac{A+C}{2}\right)z}\Bigg]$$
$$\frac{J_1(\xi r)J_1(\xi)}{\xi}\,d\xi \tag{39}$$

$$v_p = \int_0^\infty C \left[\left(u_{10} - \frac{u_0}{T_0} T_{10} - u_{p_{10}}\right) e^{-cz} + \left[(u_{p_{10}} - u_{10}) \cosh\left(\sqrt{BC + \left(\frac{A-C}{2}\right)^2} z\right) \right.\right.$$

$$\left. + \left[(u_{10} + u_{p_{10}}) \left(\frac{A-C}{2}\right) - Bu_{p_{10}} + Cu_{10} \right] \frac{\sinh\left(\sqrt{BC + \left(\frac{A-C}{2}\right)^2} z\right)}{\sqrt{BC + \left(\frac{A-C}{2}\right)^2}} \right].$$

$$e^{-\left(\frac{A+C}{2}\right)z} - C.P \frac{u_0}{T_0} (ET_{p_{10}} - (C-F)T_{10}) e^{-cz} + \frac{u_0}{T_0} \Big[T_{10} + C.P.$$

$$(ET_{10} - (C-F)T_{10}) \Big] e^{-\left(\frac{D+F}{2}\right)z} \cosh\left(\sqrt{EF + \left(\frac{D-F}{2}\right)^2} z\right) + \frac{u_0}{T_0}.$$

$$\left[\left(ET_{p_{10}} - \left(\frac{D-F}{2} + C\right) T_{10} \right) - C.P (ET_{p_{10}} - (C-F)T_{10}) \left(C - \frac{D+F}{2}\right) \right]$$

$$e^{-\left(\frac{D+F}{2}\right)z} \sinh \frac{\left(\sqrt{EF + \left(\frac{D-F}{2}\right)^2} z\right)}{\sqrt{EF + \left(\frac{D-F}{2}\right)^2}} \Bigg] \frac{J_1(\xi r) J_1(\xi) d\xi}{\xi} \tag{40}$$

$$\rho_p = \int_0^\infty \left[\rho_{p_{10}} + \frac{\rho_{p0}}{u_{p0}} (1 - e^{-cz}) \left(u_{p0} + \frac{u_0}{T_0} - u_{10}\right) - c \frac{u_0}{T_0} \frac{\rho_{p0}}{u_{p0}} \right.$$

$$\left[\left(T_{10} + ((F-C)T_{10} + ET_{p_{10}}) P \left(C - \frac{D+F}{2} \right) \right) e^{-\left(\frac{D+F}{2}\right)z}. \right.$$

$$\frac{\sinh\left(\sqrt{EF + \left(\frac{D-F}{2}\right)^2} z\right)}{\sqrt{EF + \left(\frac{D-F}{2}\right)^2}} + P(ET_{p_{10}} + (F-C)T_{10})(e^{-cz}$$

$$\left. \left. - e^{-\left(\frac{D+F}{2}\right)z} \cosh\left(\sqrt{EF + \left(\frac{D-F}{2}\right)^2} z\right) \right) \right] \right] J_1(\xi)\, J_0(\xi r) d\xi \tag{41}$$

where $D = \frac{1}{Pr.u_0} \left(\frac{2}{3}\alpha\rho_{p0} + \xi^2 T_0^{3/2}\right)$,

$E = \frac{2}{3} \frac{a}{Pr} \frac{\rho_{p0}}{u_0}$, $\quad F = \frac{2}{3} \frac{1}{Pr.u_{p0}} \frac{c_p}{c_s}$,

$P = \frac{1}{C^2 + FD - C(D+F) - EF}$ and J_0, J_1 are the Bessel functions of zeroth and first order of the first kind respectively.

4. *Discussion*

To get an insight into the interaction of the dust particles with the fluid, numerical computations have been made. The dimensionless velocity, temperature and density at the exit of the nozzle are taken to be unity and $Pr=0{\cdot}72$, $c_s/c_p=1{\cdot}0$. $u_{10}=u_{p_{10}}=T_{10}=T_{p_{10}}=\rho_{p_{10}}=0{\cdot}1$. The results have been presented in graphs to show the influence of the concentration parameter α on the physical quantities.

To evaluate the improper integrals given by (35)–(41), Gauss three-point quadrature rule have been employed. To avoid the singularity $\xi=0$, these integrals have been evaluated taking lower limit to be small enough ($\xi=0{\cdot}0625$) and upper limit to be $\xi=20$ for obtaining sufficient convergency. However, asymptotic values of $J_o(\xi)$ and $J_1(\xi)$ have been considered for large values of the argument.

Fig. 1 shows the axial velocity of the fluid and the particle phase for $\alpha=0{\cdot}1$, $0{\cdot}2$ and for $z=0{\cdot}0625$, $0{\cdot}25$. It is observed that the axial fluid velocity u is greater than the axial particle velocity u_p near the nozzle exit

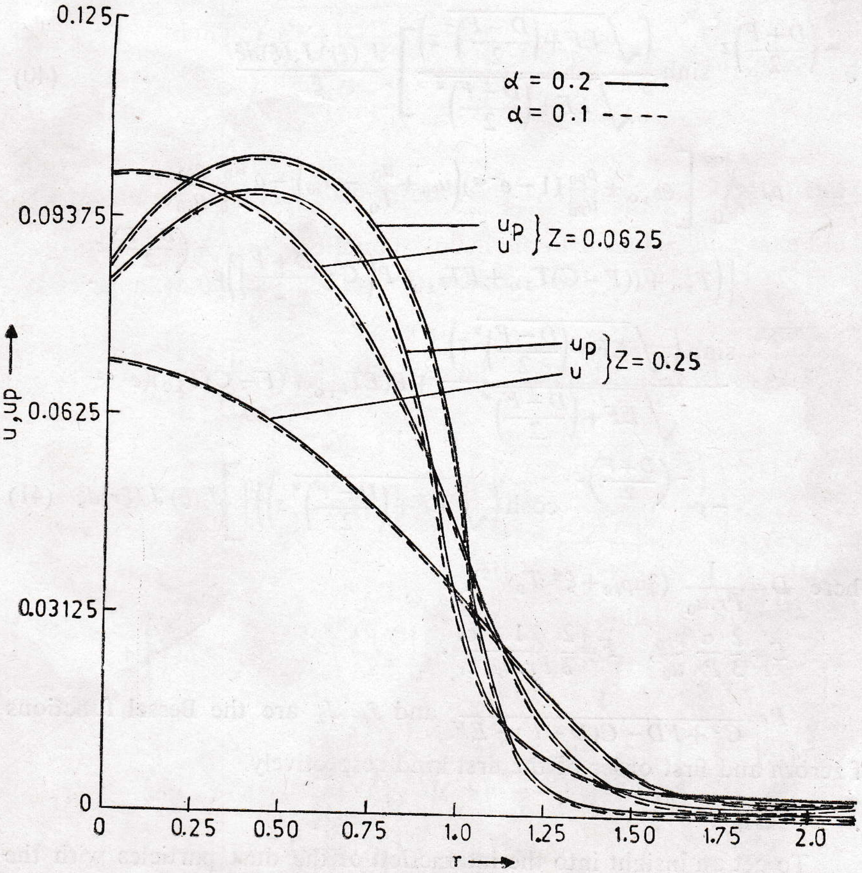

Fig. 1 : Profiles of fluid and particle perturbation velocity

whereas u_p is greater in the mixing region and finally both the velocities attain the free stream value at far downstream. The zeros of u and u_p are shifted as we move in the downstream direction, indicating gradual broadening of the jet width. The magnitude of u and u_p increases with the increase of dust parameter α.

The profiles of the perturbation fluid and particle temperatures T and T_p respectively have been shown in Fig. 2. The thermal interaction between the fluid and particle phase indicates that heat is transferred from particle to the gas in the mixing region of the jet. Therefore, a compressible dusty gas jet does not cool as fast as a pure compressible gas jet. With the increase of dust parameter α the magnitudes of T and T_p increase.

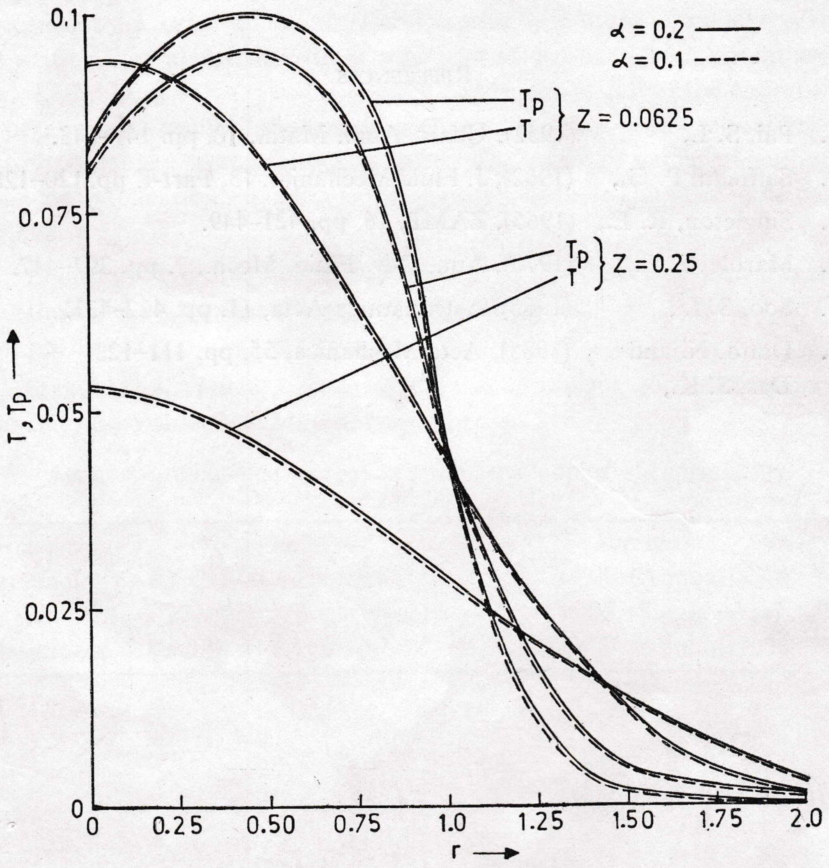

Fig. 2 : Profiles of fluid and particle perturbation temperature

In comparison with the results of the incompressible flow [c.f. 6] it is found that the jet width does not change significantly. But the fluid and particle temperatures are greater in the present case due to the compressibility nature of the fluid. Further, the present result compares favourably with the numerical results of the clear fluid given by Pai [1].

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