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EKMAN LAYER ON A POROUS PLATE WITH TIME DEPENDENT SUCTION OR INJECTION

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ABSTRACT

Hydrodynamic boundary layer flows in a semi-infinite expansion of a rotating viscous fluid bounded by an infinite porous flat plate with time dependent suction or injection is studied. We have considered two different cases, viz., (i) the plate velocity changes impulsively and (ii) the plate velocity changes in an accelerated manner. It is found that the non-dimensional shear stresses due to the unsteady primary and secondary flows, for both the cases, increase with the increase in rotation parameter while they decrease with increase in variable suction parameter.

1. *Introduction* :

From the physical point of view, the motion of a viscous incompressible fluid in a rotating frame of reference has considerable interest in many cosmical and geophysical fluid dynamics. Recently, Gupta [1] studied the steady flow of a viscous incompressible fluid past an infinite porous flat plate in a rotating frame of reference. The fluctuating flow of a viscous fluid past an infinite porous plate has been studied by Puri [2]. The boundary layer flow past a porous flat plate with variable suction or blowing has been studied by Pop and Soundalgekar [3] and Debnath and Sengupta [4]. They have considered the periodic suction velocity with non-torsional plate velocity.

In the present paper, we have considered the viscous incompressible fluid past an infinite porous plate with variable suction or blowing at the plate. We assume that both the suction velocity and the plate velocity are arbitrary functions of time. It is found that the non-dimensional shear stress components first increase, reach a maximum and then decrease with increase in rotation parameter, while they decrease with variable suction parameter.

2. *Mathematical Analysis* :

Consider an infinite porous plate coinciding with the plane $z=0$ rotating in unison with a viscous incompressible liquid occupying the region $z>0$ with a uniform angular velocity Ω about z-axis. The plate is moving with a velocity $U(t)$ along x-axis. The y-axis normal to the xz-plane. The horizontal homogeneity of the problem demands, that conditions depend on z and t only. It is evident from the equation of continuity that w is only a function of time.

We consider w as

$$w=-w_0[1+\epsilon A\, G(\tau)] \tag{1}$$

where w_0 is the constant part of the suction velocity and $G(\tau)$ is an arbitrary function of time, ϵ is a small value and A is a real positive constant such that $\epsilon A<<1$. Then the equations of motion along x and y directions are

$$\frac{\partial u}{\partial t}+w(t)\frac{\partial u}{\partial z}-2\Omega v=\nu\frac{\partial^2 u}{\partial z^2} \tag{2}$$

$$\frac{\partial v}{\partial t}+w(t)\frac{\partial v}{\partial z}+2\Omega u=\nu\frac{\partial^2 v}{\partial z^2} \tag{3}$$

where ν is the kinematic coefficient of viscosity and $w(t)$ is given by (1).

The boundary conditions are

$$u=U(t),\ v=0 \text{ at } z=0 \text{ and } u\to 0,\ v\to 0 \text{ as } z\to\infty \tag{4}$$

Assuming the plate velocity $U(t)$ in the form

$$U(t)=U_0[-1+\epsilon F(\tau)] \tag{5}$$

we get

$$u=U_0[u_0(\eta)+\epsilon u_1(\eta,\tau)],\quad v=U_0[v_0(\eta)+\epsilon v_1(\eta,\tau)] \tag{6}$$

where

$$\eta=U_0 z/\nu \text{ and } \tau=U_0^2 t/\nu \tag{7}$$

Substituting (1) and (6) in equations (2) and (3) and equating steady and unsteady parts we get

$$\frac{d^2 f_0}{d\eta^2}+S\frac{df_0}{d\eta}-2ik^2 f_0=0 \tag{8}$$

$$\frac{\partial f_1}{\partial \tau}-S\frac{\partial f_1}{\partial \eta}+2ik^2 f_1=\frac{\partial^2 f_1}{\partial \eta^2}+SAG(\tau)\frac{\partial f_0}{\partial \eta} \tag{9}$$

where

$$f_0=u_0+iv_0 \text{ and } f_1=u_1+iv_1 \tag{10}$$

In the above equations (8) and (9), $S=w_0/U_0$ is the suction parameter and $k^2=\Omega\nu/U_0^2$ is the rotation parameter.

The boundary conditions are

$$f_0=-1 \text{ at } \eta=0,\ f_0\to 0 \text{ as } \eta\to\infty \tag{11}$$

$$f_1=F(\tau) \text{ at } \eta=0,\ f_1\to 0 \text{ as } \eta\to\infty \tag{12}$$

The solution of the equation (8) subject to the boundary conditions (11) is

$$f_0=-\exp\left\{-\left(\frac{S}{2}+\alpha+i\beta\right)\eta\right\} \tag{13}$$

where

$$\alpha,\ \beta=\frac{1}{\sqrt{2}}\left[\left(\frac{S^4}{16}+4k^4\right)^{1/2}\pm\frac{S^2}{4}\right]^{1/2} \tag{14}$$

Using Laplace transform technique, equation (9) becomes

$$\frac{d^2 f_1^*}{d\eta^2}+S\frac{df_1^*}{d\eta}-(p+2ik^2)f_1^*=-SAG^*\frac{df_0}{d\eta} \tag{15}$$

where

$$f_1^*=\int_{-\infty}^{\infty} f_1(\eta,\tau)e^{-p\tau}d\tau \quad (p>0) \tag{16}$$

provided

$$f_1 e^{-p\tau}\to 0 \text{ as } \tau\to\pm\infty$$

The boundary conditions (12) become

$$f_1^*=F^* \text{ at } \eta=0 \text{ and } f_1^*\to 0 \text{ as } \eta\to\infty \tag{17}$$

Using (13), the solution of (15) subject to the boundary conditions (17) is

$$f_1^*(\eta,p)=e^{-\frac{S}{2}\eta}(\phi_1^*+\phi_2^*) \tag{18}$$

where

$$\phi_1^*=F^*\exp\{-(a+p)^{1/2}\eta\} \tag{19}$$

$$\phi_2^*=SA\left(\frac{S}{2}+\alpha+i\beta\right)\frac{G^*}{p}\left\{e^{-(\alpha+i\beta)\eta}-e^{-(a+p)^{1/2}\eta}\right\} \tag{20}$$

$$a=\tfrac{1}{4}S^2+2ik^2 \tag{21}$$

Taking inverse Laplace transform, the above equations become

$$f_1=e^{-\frac{S}{2}\eta}(\phi_1+\phi_2) \tag{22}$$

where

$$\phi_1=\frac{\eta}{2\sqrt{\pi}}\int_0^\infty F(\tau-\lambda)\lambda^{-3/2}e^{-(a\lambda+\eta^2/4\lambda)}\,d\lambda \tag{23}$$

$$\phi_2=SA\left(\frac{S}{2}+\alpha+i\beta\right)\int_0^\infty G(\tau-\lambda)\Big[e^{-(\alpha+i\beta)\eta} -\tfrac{1}{2}\Big\{e^{\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\lambda}+\sqrt{a\lambda}) +e^{-\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\lambda}-\sqrt{a\lambda})\Big\}\Big]d\lambda \tag{24}$$

We shall now discuss some particular cases corresponding to various forms of $F(\tau)$ and $G(\tau)$.

Case 1— The plate velocity changes impulsively along with its suction velocity.

Here $F(\tau)=G(\tau)=\triangle H(\tau)$, where $H(\tau)$ is the Heaviside unit function defined by

$$H(\tau)=0,\ \tau<0$$
$$H(\tau)=1,\ \tau>0 \tag{25}$$

where $\triangle$ is a constant.

Substituting the above value of $F(\tau)$ and $G(\tau)$ in the equations (23) and (24) and evaluating the integrals, we get

$$\phi_1=\tfrac{1}{2}\triangle H(\tau)\Big[e^{\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\tau}+\sqrt{a\tau}) +e^{-\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\tau}-\sqrt{a\tau})\Big] \tag{26}$$

$$\phi_2=\tfrac{1}{2}SA\triangle H(\tau)\left(\frac{S}{2}+\alpha+i\beta\right)\Big[2\tau e^{-(\alpha+i\beta)\eta}-\Big\{(\tau+\eta/2\sqrt{a})\,e^{\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\tau}+\sqrt{a\tau})+(\tau-\eta/2\sqrt{a})\,e^{-\sqrt{a}\eta}\ \mathrm{erfc}\ (\eta/2\sqrt{\tau}-\sqrt{a\tau})\Big\}\Big] \tag{27}$$

Substituting ϕ_1 and ϕ_2 in the equation (22) and using the definition of f_1 given by (10) we get u_1 and v_1

The non-dimensional shear stress at the plate $\eta=0$ is given by

$$\tau_{xi}+i\tau_{yi}=\triangle H(\tau)\Big[-\Big\{\frac{S}{2}+\sqrt{a}\ \mathrm{erf}\ (\sqrt{a\tau})+\frac{1}{\sqrt{\pi\tau}}\ e^{-a\tau}\Big\}$$
$$+SA\Big(\frac{S}{2}+\alpha+i\beta\Big)\Big\{-\tau(\alpha+i\beta)\Big(\sqrt{a\tau}+\frac{1}{2\sqrt{a}}\Big)\ \mathrm{erf}\ (\sqrt{a\tau})$$
$$+\Big(\frac{\tau}{\pi}\Big)^{1/2}\ e^{-a\tau}\Big\}\Big] \qquad (28)$$

On separating real and imaginary parts we get τ_{xi} and τ_{yi} at the plate $\eta=0$. The values of τ_{xi} and τ_{yi} have been plotted against τ for different values of rotation parameter k^2 in Fig. 1. It is found that both τ_{xi} and τ_{yi} increase with increase in either k^2 or τ when A is constant.

Table 1 shows that for fixed k^2 and τ, both τ_{xi} and τ_{yi} decrease with increase in A.

For small time

$$\phi_1=\triangle H(\tau)\Big[\mathrm{erfc}\ (\eta/2\sqrt{\tau})-\Big(\frac{S^2}{4}+2ik^2\Big)\Big\{\Big(\frac{\tau}{\pi}\Big)^{1/2}\ e^{-\eta^2/4\tau}$$
$$-\tfrac{1}{2}\eta\ \mathrm{erfc}\ (\eta/2\sqrt{\tau})\Big\}\Big] \qquad (29)$$

$$\phi_2=SA\triangle H(\tau)\Big(\frac{S}{2}+\alpha+i\beta\Big)\Big[\tau\, e^{-(\alpha+i\beta)\eta}-\Big(\tau+\frac{1}{2}\eta^2\Big)\ \mathrm{erfc}\ (\eta/2\sqrt{\tau})$$
$$+\eta\Big(\frac{\tau}{\pi}\Big)^{1/2} e^{-\eta^2/4\tau}+\Big(\frac{S^2}{4}+2ik^2\Big)\Big\{\Big(\frac{\tau}{\pi}\Big)^{1/2}\Big(\frac{2}{3}\tau+\frac{1}{6}\eta^2\Big)$$
$$\times e^{-\eta^2/4\tau}-\frac{1}{2}\eta\Big(\tau+\frac{1}{6}\eta^2\Big)\ \mathrm{erfc}\ (\eta/2\sqrt{\tau})\Big\}\Big] \qquad (30)$$

Substituting (29) and (30) in (22) and using (10) we get u_1 and v_1 for small time. It is interesting to note that the unsteady primary velocity u_1 depends on the rotation parameter if we considered the variable suction. For constant suction, u_1 is independent of rotation parameter k^2 while the unsteady secondary velocity v_1 depends on rotation parameter for both variable as well as constant suction.

For large time u_1 and v_1 become

$$u_1=\triangle H(\tau)[e^{-(\frac{S}{2}+\alpha)\eta}\cos\beta\eta-2\eta\Big(\frac{\tau}{\pi}\Big)^{1/2}\{(S^2\tau^2-\eta^2)\cos 2k^2\tau$$
$$-8k^2\tau^2\sin 2k^2\tau\}\,e^{-(S\tau+\eta)^2/4}+\frac{SA}{2(\alpha^2+\beta^2)}\,\eta e^{-(\frac{S}{2}+\alpha)\eta}$$
$$\cdot\Big\{\Big(\frac{S}{2}+\alpha\Big)(\alpha\cos\beta\eta-\beta\sin\beta\eta)+\beta(\beta\cos\beta\eta+\alpha\sin\beta\eta)\Big\}] \qquad (31)$$

$$v_1=\triangle H(\tau)[e^{-(\frac{S}{2}+\alpha)\eta}\{-\sin\beta\eta+\frac{SA}{2(\alpha^2+\beta^2)}[\beta(\alpha\cos\beta\eta$$

$$-\beta\sin\beta\eta)-\left(\frac{S}{2}+\alpha\right)(\beta\cos\beta\eta+\alpha\sin\beta\eta)]+2\eta\left(\frac{\tau}{\pi}\right)^{1/2}$$

$$\cdot\{(S^2\tau^2-\eta^2)\sin 2k^2\tau+8k^2\tau^2\cos 2k^2\tau\}e^{-(S\tau+\eta)^2/4} \tag{32}$$

The above equations show that both the primary and the secondary velocities are the combination of a steady state solution and the unsteady solution. The steady state is confined within a thin layer. The thickness of this layer is $O(2/(S+2\alpha))$. Since α (see eqn. (14) increases with increase in either $S(S>0)$ or k^2 while it decreases with increase in injection $S(S<0)$. This implies that boundary layer thickness decreases with increase in either $S(S>0)$ or k^2 but it increases with increase in $S(S<0)$. The unsteady solution represents the inertial oscillations of the fluid which ultimately die out for large time. The frequency of these oscillations is $2k^2$, which is independent of both suction as well as injection.

Case 2—The plate starts with sudden acceleration and its suction velocity changes impulsively.

In this case we put $F(\tau)=\triangle\tau H(\tau)$ and $G(\tau)=\triangle H(\tau)$ in (23) and (24), we get

$$\phi_1=\tfrac{1}{2}\triangle H(\tau)[(\tau+\eta/2\sqrt{a})e^{\sqrt{a}\eta}\operatorname{erfc}(\eta/2\sqrt{\tau}+\sqrt{a\tau})$$

$$+(\tau-\eta/2\sqrt{a})\,e^{-\sqrt{a}\eta}\operatorname{erfc}(\eta/2\sqrt{\tau}-\sqrt{a\tau})] \tag{33}$$

and ϕ_2 is given by (27). Knowing ϕ_1 and ϕ_2 one can easily obtain $f_1(\eta,\tau)$ from equation (22) and separating real and imaginary parts we get u_1 and v_1.

The non-dimensional shear stresses due to the unsteady primary and the secondary flow at the plate $\eta=0$ are

$$\tau_{xa}+i\tau_{ya}=\triangle H(\tau)\Big[-\left(\frac{S}{2}+\alpha+i\beta\right)SA(\alpha+i\beta)\tau-\frac{S}{2}\tau$$

$$+\left\{\left(\frac{S}{2}+\alpha+i\beta\right)-1\right\}\left\{(\tau\sqrt{a}+1/2\sqrt{a})\operatorname{erf}(\sqrt{a\tau})+\left(\frac{\tau}{\pi}\right)^{1/2}e^{-a\tau}\right\}\Big] \tag{34}$$

The values of τ_{ya} and τ_{xa} have been plotted against τ for different values of k^2 in Fig. 2. It is seen that both τ_{xa} and τ_{ya} increase with increase in k^2 or τ. It is also seen from Table 2 that they decrease with increase in A.

For small time ϕ_1 is given by

$$\phi_1 = \triangle H(\tau)\Big[(\tau+\tfrac{1}{2}\eta^2)\,\mathrm{erfc}\,(\eta/2\sqrt{\tau})-\eta\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\eta^2/4\tau}$$

$$-a\eta\left\{\left(\frac{\tau}{\pi}\right)^{1/2}(\tfrac{2}{3}\tau+\tfrac{1}{6}\eta^2)\,e^{-\eta^2/4\tau}-\tfrac{1}{2}\eta(\tau+\tfrac{1}{6}\eta^2)\,\mathrm{erfc}\,(\eta/2\sqrt{\tau})\right\}\Big] \tag{35}$$

and ϕ_2 is given by (30).

Substituting (35) and (30) in (22) and on using $f_1(\eta, \tau)$ as in (10) we get u_1 and v_1 for small time. As in Case 1 here the unsteady primary velocity depends on rotation only, when variable suction velocity is taken into account.

For large time, u_1 and v_1 are given by

$$u_1 = \triangle H(\tau)\Big[\tau\cos\beta\eta-\frac{\eta}{2(\alpha^2+\beta^2)}(\alpha\cos\beta\eta-\beta\sin\beta\eta)\,e^{-(\frac{S}{2}+\alpha)\eta}$$

$$+\frac{SA}{2(\alpha^2+\beta^2)}\eta e^{-(\frac{S}{2}+\alpha)\eta}\left\{\left(\frac{S}{2}+\alpha\right)(\alpha\cos\beta\eta-\beta\sin\beta\eta)\right.$$

$$\left.+\beta(\beta\cos\beta\eta+\alpha\sin\beta\eta)\right\}\Big] \tag{36}$$

$$v_1 = \triangle H(\tau)\Big[-\tau\sin\beta\eta+\frac{\eta}{2(\alpha^2+\beta^2)}(\beta\cos\beta\eta+\alpha\sin\beta\eta)\,e^{-(\frac{S}{2}+\alpha)\eta}$$

$$+\frac{SA}{2(\alpha^2+\beta^2)}\eta\left\{\beta(\alpha\cos\beta\eta-\beta\sin\beta\eta)-\left(\alpha+\frac{S}{2}\right)\right.$$

$$\left.\cdot(\beta\cos\beta\eta+\alpha\sin\beta\eta)\right\}e^{-(\frac{S}{2}+\alpha)\eta}\Big] \tag{37}$$

It is interesting to note that when the plate starts with sudden acceleration and the suction velocity changes impulsively then there is no inertial oscillation.

Table 1

Values of τ_{xi} and τ_{yi} for $k^2=2{\cdot}0$, $\tau=0{\cdot}5$ and $S=1{\cdot}0$

A	0·0	0·2	0·4	0·6	0·8
$-\tau_{xi}$	1·86935	1·78832	1·70700	1·62628	1·54526
$-\tau_{yi}$	2·00147	1·86953	1·73758	1·60564	1·47369

Table 2

Values of τ_{xa} and τ_{ya} for $k^2=2{\cdot}0$, $\tau=0{\cdot}5$ and $S=1{\cdot}0$

A	0·0	0·2	0·4	0·6	0·8
$-\tau_{xa}$	1·27654	1·19552	1·11450	1·03340	0·95245
$-\tau_{ya}$	0·81430	0·68236	0·55041	0·41847	0·28652

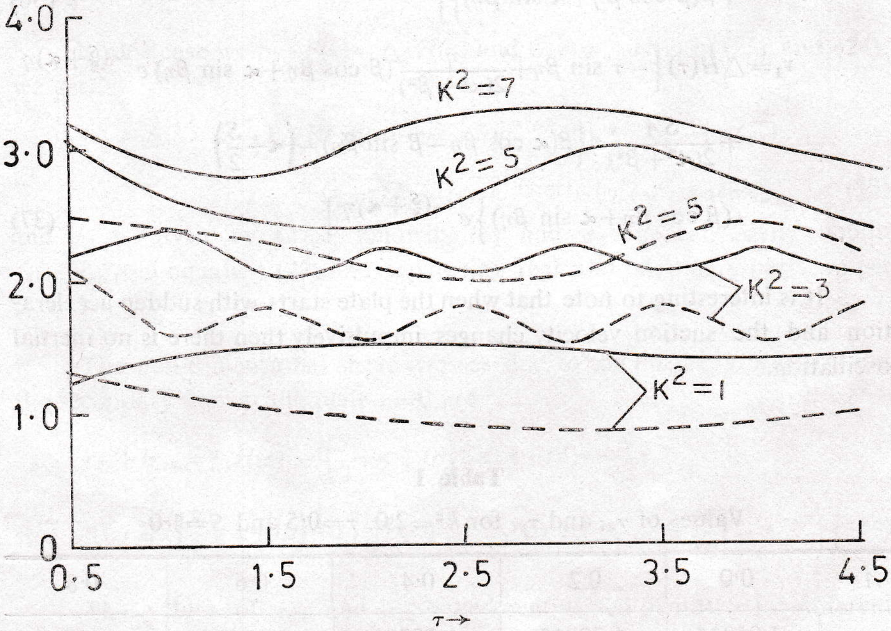

Fig. 1 Shear stress components for $A=0{\cdot}2$

——— τ_{xi} and — — — — τ_{yi}

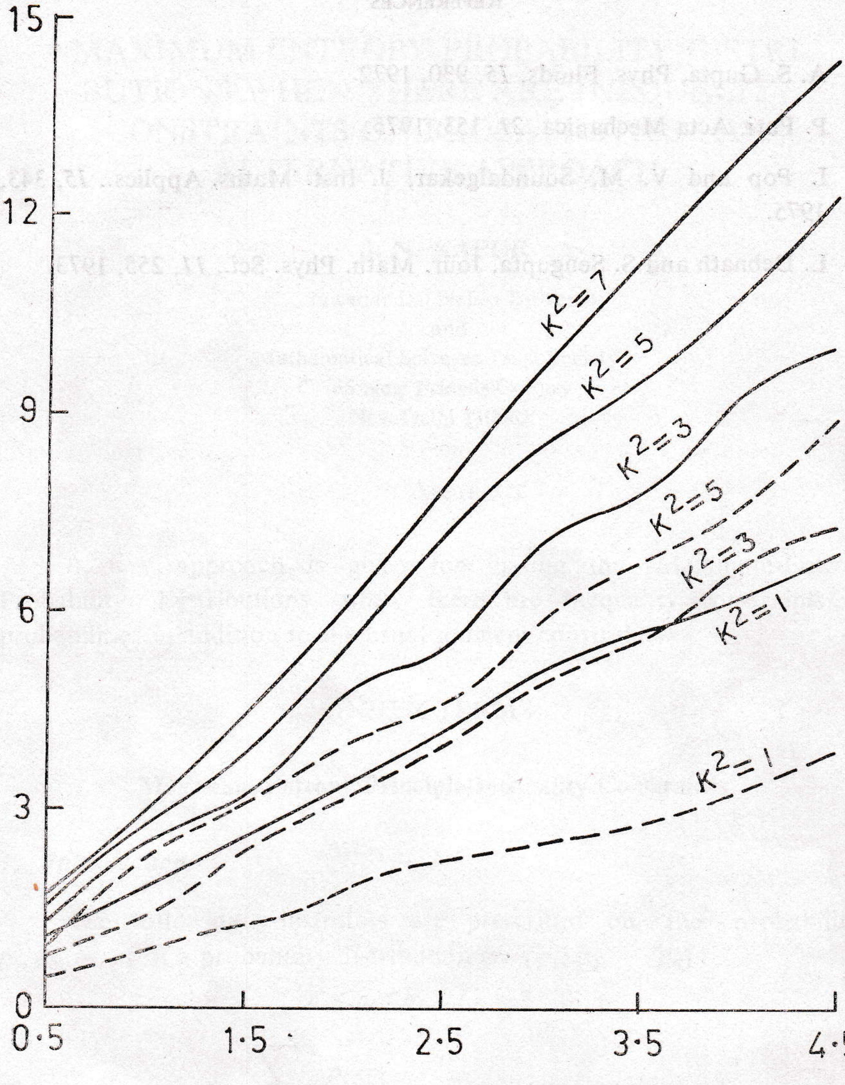

Fig. 2 Shear stress components against τ for $A=0{\cdot}2$

—— τ_{xa} and – – – τ_{ya}

References

1. A. S. Gupta, Phys. Fluids, *15*, 930, 1972.
2. P. Puri, Acta Mechanica, *21*, 153, 1975.
3. I. Pop and V. M. Soundalgekar, J. Inst. Maths. Applics., *15*, 343, 1975.
4. L. Debnath and S. Sengupta, Jour. Math. Phys. Sci., *11*, 255, 1973.