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FULLY DEVELOPED LAMINAR FLOW IN A POROUS RECTANGULAR DUCT

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ABSTRACT

The fully developed laminar flow of a viscous incompressible liquid in a long porous straight duct of rectangular cross-section has been investigated. The generalised momentum equation which governs the flow has been solved by using Fourier's transform. Expressions for velocity distribution and volume flow rate (flux) have been derived. It is found that the velocity and the volume flow rate increase with the increase of the permeability of the porous medium, whereas the flux decreases with the decrease in the value of the aspect ratio.

1. *Introduction* :

Fluid flow through porous media plays important role in hydrology, petroleum engineering, chemical engineering, bio-chemical engineering, agricultural engineering, medicines and paper technology, etc. Henry Darcy (1856) initiated the mathematical theory of flow of a fluid through porous medium. Darcy's law states that the seepage velocity of the fluid is proportional to the pressure gradient. Francher, Lewis and Barnes (1933) demonstrated experimentally that the flow through sands remains laminar and the Darcy's law is valid so long as the Reynolds number is equal to or less than unity. For slowly spatially varying flow, Darcy's law is surely correct. If, however, the flow involves large shear, one expects further terms involving velocity gradients to appear. Brinkman (1947) put Darcy's law in a better theoretical shape by taking into account the effect of viscous stress. This generalised Darcy's law gave good results in the case of highly porous media.

The problem of fully developed flow through triangular, square and rectangular shaped ducts filled with porous media or in absence of it has been studied by many scientists—Ramachandra and Spalding (1978), Wong (1979), Iqbal and Aggarwala (1971), Patil (1979). Rahman (1983) has studied numerically the problem of fully developed laminar flow in a rectangular duct. This problem has been studied analytically by Borkakati and Rahman (1984). They have obtained the expressions for shearing stresses which vanish at the corners of the channel.

In this paper, we have studied the problem of fully developed laminar flow of an incompressible viscous fluid in a long porous rectangular duct by using the generalised Darcy's law.

2. *Formulation and solution of the problem* :

We consider a rectangular duct of width a and length b filled with a porous material of permeability K. A Cartesian coordinate system is used in such a way that the walls of the rectangle be at $X=0$, $X=a$, $Y=0$ and $Y=b$, and the flow of the incompressible viscous fluid inside the duct is assumed to be fully developed. All the four boundaries are impermeable to fluid flow. Permeability K of the porous medium and all fluid properties are taken to be constant. The velocity components in the directions of X, Y and Z are assumed to be of the form

$$U=0,\ V=0 \text{ and } W=W\ (X,\ Y) \tag{1}$$

respectively.

The equation of continuity

$$\nabla \cdot \vec{V}=0 \tag{2}$$

is satisfied identically by the form of velocity components (1). The equation of motion of a viscous fluid through a porous medium as proposed by Brinkman (1947) is

$$\nabla P=\mu \nabla^2 \vec{V}-\frac{\mu}{K}\vec{V} \tag{3}$$

where $\vec{V}$ is the velocity vector, μ is the viscocity of the fluid and K is the permeability constant of the medium.

The equation of motion (3), for the velocity components of the form (1), becomes

$$\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}-\frac{W}{K}=\frac{1}{\mu}\frac{dP}{dZ} \tag{4}$$

where $\frac{dP}{dZ}$ is the constant pressure gradient.

The boundary conditions of the problem are that the velocity component W is zero at all the four faces of the rectangular duct. Hence

$$W(0, Y)=W(a, Y)=W(X, 0)=W(X, b)=0 \tag{5}$$

Introducing the dimensionless variables defined by

$$X=ax, \; Y=by, \; W=W_1 w \tag{6}$$

where

$$W_1=a^2\left(-\frac{1}{\mu}\frac{dP}{dZ}\right) \tag{7}$$

we obtain the transformed equation with boundary conditions as

$$\frac{\partial^2 w}{\partial x^2}+\alpha^2\frac{\partial^2 w}{\partial y^2}-\delta^2 w+1=0 \tag{8}$$

and

$$w(0, y)=w(1, y)=w(x, 0)=w(x, 1)=0 \tag{9}$$

where $\alpha=a/b$ is the aspect ratio and $\delta=a/\sqrt{K}$ is the permeability parameter.

Using finite Fourier sine transform, we get the solution of (8) subjected to the boundary conditions (9). The expression for velocity is found to be

$$w(x, y)=\frac{4}{\pi}\sum_{n=0}^{\infty}\left[\frac{1}{\alpha^2 m^2}\left\{1-\frac{\sinh my+\sinh m(1-y)}{\sinh m}\right\}\frac{\sin(2n+1)\pi x}{2n+1}\right] \tag{10}$$

where

$$m^2=\frac{\delta^2+\pi^2(2n+1)^2}{\alpha^2}$$

3. *Results and discussion* :

The expression (10) for velocity w is symmetric at $x=0{\cdot}5$ and $y=0{\cdot}5$. The numerical results of w for x, $y=0$, $0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$, $0{\cdot}5$ when $\alpha=0{\cdot}5$, 1 and $\delta=1$, 2 are shown in Table 1. From the results it is observed that w increases when either x or y increases from 0 to $0{\cdot}5$ while keeping other variables constant. The velocity w decreases with the decrease of the permeability of the porous medium and with the increase of the length a. The velocity w attains its maximum value at the line $x=0{\cdot}5$, $y=0{\cdot}5$ which compares favourably with the results given by Borkakati and Rahman (1984).

The volume flow rate

$$Q=\int_0^a\int_0^b W\,dy\,dx$$

can be written in dimensionless form as

$$\frac{Q}{abW_1}=\bar{Q}=\int_0^1\int_0^1 w\,dy\,dx$$

i.e.
$$\bar{Q}=\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{\alpha^2 m^2(2n+1)^2}\left[1-\frac{2}{m}\tanh\frac{m}{2}\right] \tag{11}$$

The graph of flux $\bar{Q}$ is plotted against the permeability parameter δ for various values of the aspect ratio α in Fig. 1. It is observed from the figure that the flux decreases with the increase of the permeability parameter. The flux decreases with the decrease in the value of the aspect ratio.

$y \backslash x$	0	0·1	0·2	0·3	0·4	0·5	α	δ
0	0	0	0	0	0	0		
0·1	0	0·0219	0·0365	0·0460	0·0513	0·0530	0	
0·2	0	0·0314	0·0541	0·0695	0·0785	0·0814		
0·3	0	0·0360	0·0627	0·0813	0·0923	0·0958	0·5	1
0·4	0	0·0381	0·0667	0·0868	0·987	0·1025		
0·5	0	0·0387	0·0678	0·0884	0·1005	0·1045		
0	0	0	0	0	0	0		
0·1	0	0·0193	0·0316	0·0393	0·0463	0·0485		
0·2	0	0·0269	0·0455	0·0579	0·0649	0·0671		
0·3	0	0·0302	0·0518	0·0665	0·0749	0·0776	0·5	2
0·4	0	0·0316	0·0545	0·0702	0·0793	0·0822		
0·5	0	0·0320	0·0553	0·0719	0·0805	0·0835		
0	0	0	0	0	0	0		
0·1	0	0·0127	0·0202	0·0247	0·0271	0·0279		
0·2	0	0·0202	0·0332	0·0414	0·0459	0·0473		
0·3	0	0·0247	0·0414	0·0522	0·0582	0·0602	1	1
0·4	0	0·0271	0·0459	0·0582	0·0652	0·0675		
0·5	0	0·0279	0·0473	0·0602	0·0675	0·0698		

Table I The value of w at various points in the duct

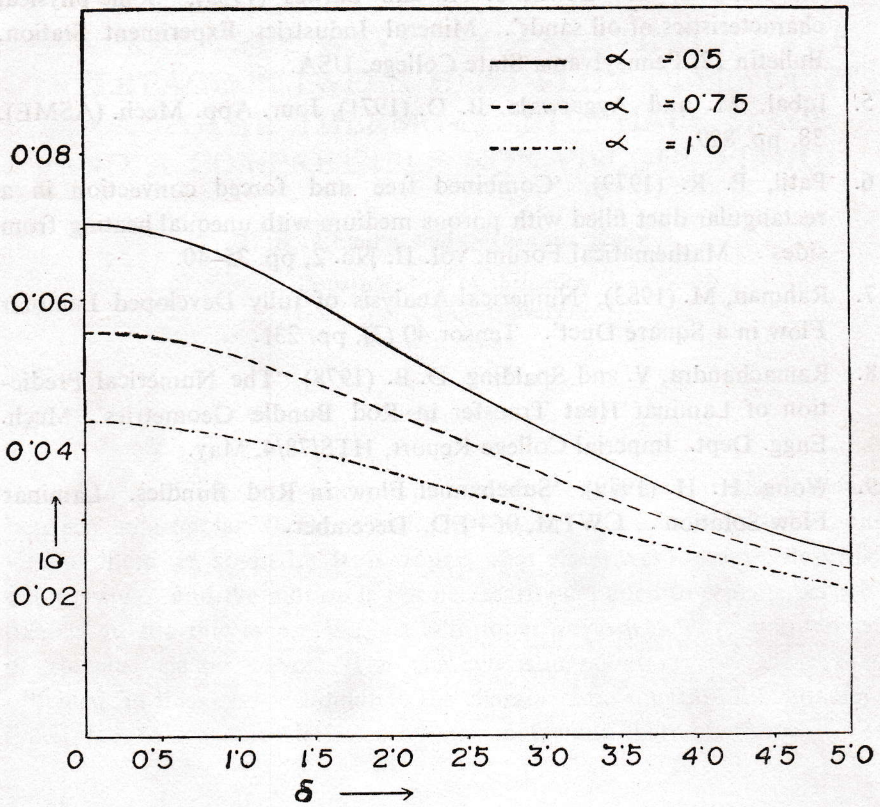

Fig. 1 The graph of flux ($\bar{Q}$) against the permeability parameter (δ)

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