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A COMPARATIVE STUDY ON FRACTIONAL DERIVATIVE MASKS FOR EDGE DETECTION ANALYSIS

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Abstract

Quantitative design and performance of edge detection is an essential stage in numerous image processing applications. Due to the extra free parameter order α, fractional order based methods provide additional degree of freedom in optimizing the performance of the technique. This work presents a comparative study of fractional order edge detection with Gradient order edge detectors, when applied to two types of images, i) Linear image; ii) Non-linear image. Further the study will be extending to compare the Fractional derivative edge detectors, applied to the above mentioned images.

Keywords: Edge detection, Gradient based edge detector, Fractional order based edge detector and Figure of merit.

2010 AMS classification: 65D18, 68U10, 68U15.

1. Introduction

Edge detection can be defined as the discovery of lines that marks the limit and divides of image appearance from other places or things in a digital image. This technique help simplifies the data to be processed during any task. A digital system uses various methods to process a digital image.It often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the

Laplacian[1, 2]. Edge detectors are used in various fields like astronomy, medical science, forensic investigation, Meteorology etc.Chen *et al.*[3] proposed the edge detection technique for fingerprint identification. Again Nikolic*et al.*[4]studiedEdge detection in medical ultrasound images using Canny edge detection algorithm. Recently, Abdel-Gawad*et al.*[5]proposes an optimized edge detection technique based on a genetic algorithm.

Fractional calculus is an old branch of mathematical analysis which deals with the integration and differentiation of a function to arbitrary order [6, 7, 8].Due to various applicability of this analysis many researchers attract in towards this topic and which lead to the introduction and investigation of several fundamental works on various aspects of fractional calculus. The fractional calculus is rarely used in image processing technique. Yang *et al.*[9] in their review paper mentioned about fractional calculus as one of the method used in ten sub-fields of image processing. Li and Xie[10] studied a new medical image enhancement method that adjusts the fractional order according to the dynamic gradient feature of the entire image.Recently, Lavin-Delgado *et al.*[11] and Aboutabit[12]proposed a new edge detection mask based on Caputo-Fabrizio fractional derivative.

The remainder of this paper is organized as follows. Section 2 describes the related theories of edge detection techniques of both classical and fractionaldifferentials. Also state the performance measure technique. Section 3 presents a comparison of integer order and non-integer order edge detectors. In section 4 we have discussed Figure of Merit(FoM) value of Grunwald-Letnikov (G-L) and Riemann-Liouville (R-L) fractional edge detector. Section 5 presents the conclusions.

2. Related theories of edge detection and analysis

The method for edge detection is classified into two categories; first is gradient based and second is Laplacian based. In this paper we are using the gradient based edge detection [13]. The details of this method areexplained in the following subsection.

2.1. Gradient based edge detection

An edge, in a digital image is a significant local change in the image intensity$f(x, y)$ at the edge point.The change of function is expressed by its gradient which is defined as follows:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad (2.1)$$

The gradient magnitude $|\nabla f|$ and orientation θ can be calculated from

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \quad (2.2)$$

$$\theta = arctg\left(\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right) \quad (2.3)$$

The most used operators are [2, 13]:

Prewitt: $P_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, P_y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$

Sobel: $S_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}, S_y = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Robert: $R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

2.2. Fractional based edge detectors

2.2.1. Grünwald-Letnikovderivative edge detector

Grünwald-Letnikovderivative is originates from the first order backward difference scheme, as described by Podlubny[8]. TheGrünwald–Letnikov(G-L) fractional derivative mask [14] are simplified as follows:

The x- directional fractional mask of three columns:

$$\begin{bmatrix} 0.5(-1)^{K-1}\binom{\alpha}{K-1} & (-1)^{K-1}\binom{\alpha}{K-1} & 0.5(-1)^{K-1}\binom{\alpha}{K-1} \\ \cdots & \cdots & \cdots \\ \frac{\alpha^2-\alpha}{4} & \frac{\alpha^2-\alpha}{2} & \frac{\alpha^2-\alpha}{4} \\ 0.5 & 1 & 0.5 \end{bmatrix}$$

The y-directional fractional mask of three rows:

$$\begin{bmatrix} 0.5(-1)^{K-1}\binom{\alpha}{K-1} & \cdots & \frac{\alpha^2-\alpha}{4} & \frac{-\alpha}{2} & 0.5 \\ (-1)^{K-1}\binom{\alpha}{K-1} & \cdots & \frac{\alpha^2-\alpha}{2} & -\alpha & 1 \\ 0.5(-1)^{K-1}\binom{\alpha}{K-1} & \cdots & \frac{\alpha^2-\alpha}{4} & \frac{-\alpha}{2} & 0.5 \end{bmatrix}$$

2.2.2. Riemann-Liouville edge detector

The Riemann-Liouville(R-L) based edge detectors considered in this work are eight-fractional differential masks which are respectively on the directions of positive x-co-ordinate, negative x-co-ordinate, positive y-co-ordinate, negative y-co-ordinate, left-upper diagonal, left-lower diagonal, right-upper diagonal and right-lower diagonal, as simplified in[15] are as follows:

(a) Positive x-co-ordinate

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \cdots & 0 & 0 & 0 \\ C_{S_{-1}} & C_{S_0} & C_{S_1} & \cdots & C_{S_k} & \cdots & C_{S_{n-2}} & C_{S_{n-1}} & C_{S_n} \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Negative x-co-ordinate

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \cdots & 0 & 0 & 0 \\ C_{S_n} & C_{S_{n-1}} & C_{S_{n-2}} & \cdots & C_{S_k} & \cdots & C_{S_1} & C_{S_0} & C_{S_{-1}} \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Positive y-co-ordinate

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & C_{S_{-1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{S_{n-2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_{n-1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_n} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(d) Negative y-co-ordinate

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & C_{S_n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_{n-1}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_{n-2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{S_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{S_{-1}} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(e) Right lower-diagonal

$$\begin{bmatrix} C_{S_{-1}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & C_{S_0} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & C_{S_1} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \vdots & 0 & \ddots & \ddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & \ddots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_{n-2}} & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_{n-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_n} \end{bmatrix}$$

(f) Left-lower diagonal

$$\begin{bmatrix} C_{S_n} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & C_{S_{n-1}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & C_{S_{n-2}} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \vdots & 0 & \ddots & \ddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & \ddots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_1} & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C_{S_{-1}} \end{bmatrix}$$

(g) Right-upper diagonal

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & C_{S_n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & C_{S_{n-1}} & 0 \\ \vdots & 0 & 0 & 0 & \cdots & 0 & C_{S_{n-2}} & 0 & 0 \\ 0 & \vdots & 0 & \ddots & \ddots & \iddots & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \iddots & \ddots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & C_{S_1} & 0 & \cdots & 0 & 0 & 0 & \vdots \\ 0 & C_{S_0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_{S_{-1}} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(h) Left- lower diagonal

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & C_{S_{-1}} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & C_{S_0} & 0 \\ \vdots & 0 & 0 & 0 & \cdots & 0 & C_{S_1} & 0 & 0 \\ 0 & \vdots & 0 & \ddots & \ddots & .. & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & C_{S_k} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & .. & \ddots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & C_{S_{n-2}} & 0 & \cdots & 0 & 0 & 0 & \vdots \\ 0 & C_{S_{n-1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_{S_n} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

2.3. Edge creation

In order to obtain an edge map, after gradient operator application, it is necessary to use post-processing, such as some of the thresholding methods [13, 16], the non-maximum suppression or a three-module strategy. In this work, we will use Otsu thresholding method [17]. This thresholding algorithm returns a single intensity threshold that separate pixels into two classes, foreground and background.

Thresholding produces wide edges. To get edges with a one-pixel width it is necessary to apply a suitable thinning procedure, usually morphological algorithm [13].

2.4. Edge Detector Performance

To compare the various edge detection techniques, it is necessary to design some quantitative criteria for the edge detection performance. Pinho and Almeida[18] introduced Figure of Merit (FoM) for the edge detection performance. FoM uses ideal edge map of an image and compares it with that to the edge map obtained after applying edge detection technique, which is defined in the following:

$$\text{Figure of Merit } (FoM) = \frac{1}{\max(N_I - N_A)} \sum_{n=1}^{N_A} \frac{1}{1 + C d_n^2} \qquad (3.4)$$

where $N_I$andN_A are the number of ideal and the number of actual detected edge pixels, respectively, $d_n$is considered as the distance between the predicted edge pixel and the closest ideal edge pixel obtained from a ground truth image, and $C$is a scaling factor that is chosen as $\frac{1}{9}$-th by following Pratt's work. It is clear that $0 \leq FoM \leq 1$. The more the detected edges match the real edges, the closer the value of FoM is to 1. If $FoM = 1$, then two compared binary image (edge map) are the same. Binary image with real edges is often referred to as a ground truth. For more details regardingFoM one can refer to the book [2].

3. Comparing integer order and non-integer order edge detectors

In this section, we will discuss about edge detection and compare the performance of various edge detectors considering two different types of image: linear and non-linear image.

3.1. Comparison for Linear image

A linear image is consideredand apply classical gradient and fractional derivative edge detector to detect the edges. We consider order 1.2 and mask size 3 for both G-L and R-L edge detectors.

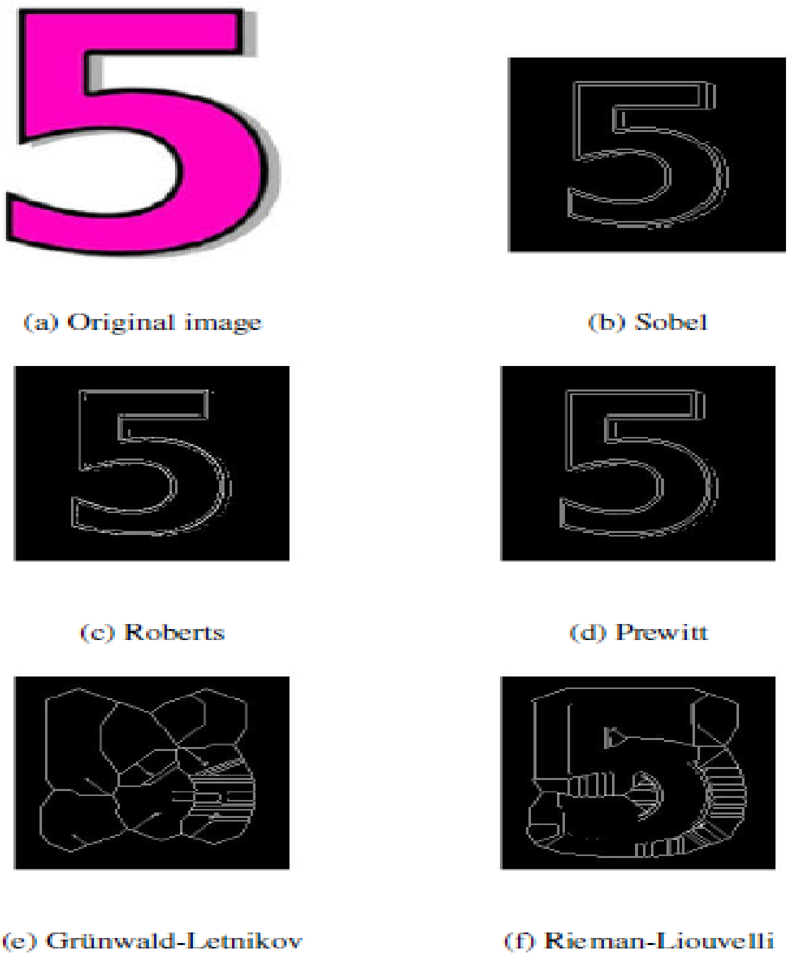


Fig.1 : Edge detection on linear image

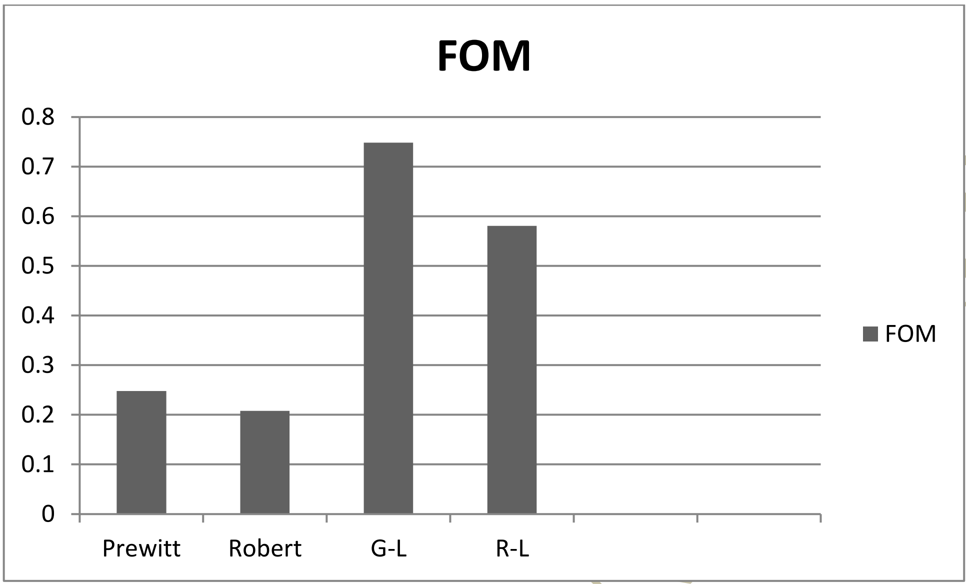


Fig.2 : Graphs for linear image.

Edge detectors	**FoM**
Sobel	0.131739
Prewitt	0.129047
Robert	0.125542
G-L	0.255758
R-L	0.345896

Table 1 : FoM values for linear image

3.2. Comparison for Non-linear image

Here we consider a non-linear image and apply classical gradient and fractional derivative edge detector to detect its edges. We consider order 1.2 and mask size 3 for both G-L and R-L edge detectors.

(a) original image

(b) Sobel

(c) Robert

(d) Prewitt

(e) Grünwald-Letnikov, order= 1.2 mask size=3

(f) Riemann-Liouvelli, order= 1.2 mask size=3

Fig.3:Edge detection of Non-linear image

Edge detectors	FoM
Sobel	0.252299
Prewitt	0.248102
Robert	0.208276
G-L	0.748263
R-L	0.580541

Table 2: FoM values of non-linear image.

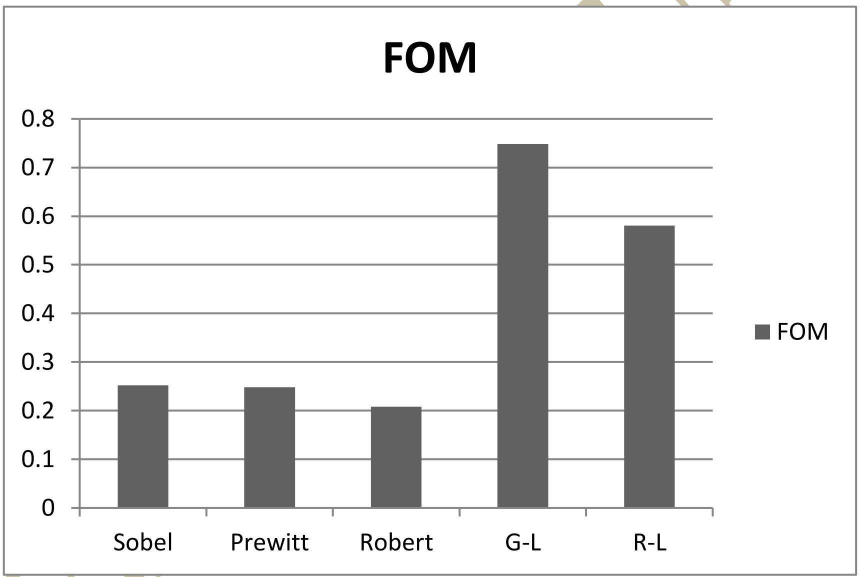


Fig.4: FoM of non-linear image

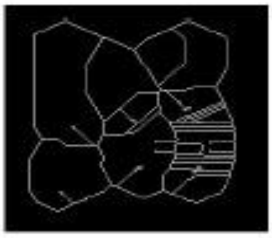
4. Comparing G-Land R-L fractional edge detector for different order

In the previous section we have observed that fractional based edge detectors produce better result in case of linear and non-linear images when compared with gradient edge detectors. Thus in this section, we calculated theFoM values of G-L and R-L fractional edge detectors for different orders i.e. $1 < \alpha < 2$, for linear and non-linear images.

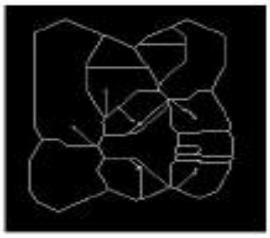
4.1. Comparing Linear image

We consider the order $\alpha = 1, 1.2, 1.4, 1.6, 1.8,\ 2$ for G-L and R-L fractional edge detectors.

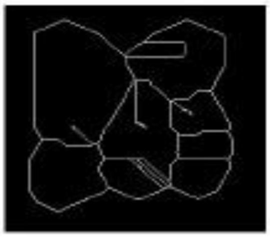
(a) order=1



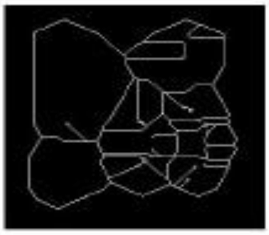
(b) order=1.2



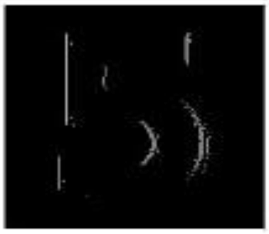
(c) order=1.4



(d) order=1.6



(e) order=1.8



(f) order=2

Fig.5:G-L edge detection of linear image.

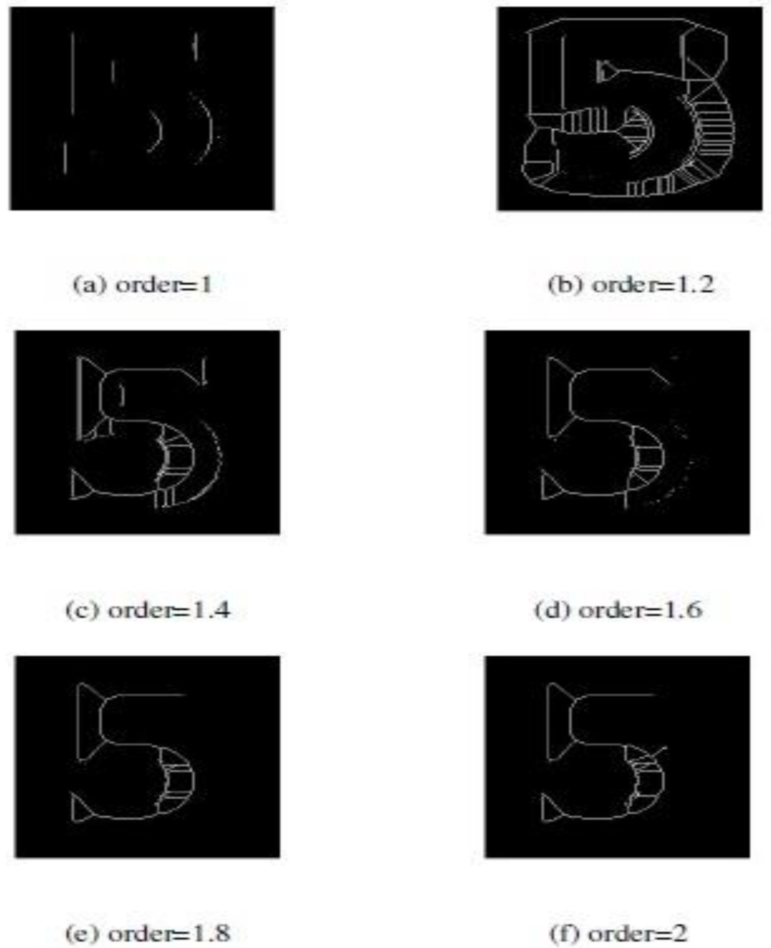


Fig.6: R-L edge detection of linear image

From the Fig.7 FoM values for order 1.2 produces better results for both the edge detectors. But as the order is tending towards 2, R-L edge detector is showing an alternate result compare with G-L edge detector. And in most of the paper in the review considers order 1.2and mask size 3.

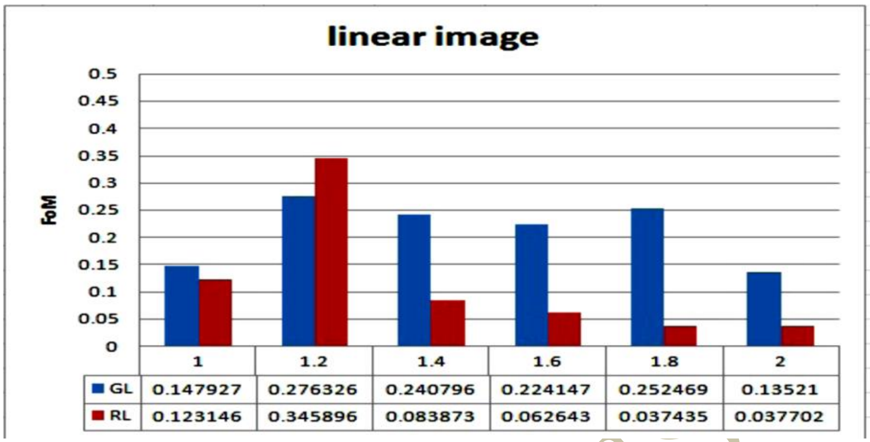


Fig.7: FoM of G-L and R-L edge detectors

4.2. Comparing non-linear image

We consider the non-linear image and apply G-L and R-L fractional order edge detection respectively for order$\alpha = 1,1.2, 1.4, 1.6, 1.8, 2$.

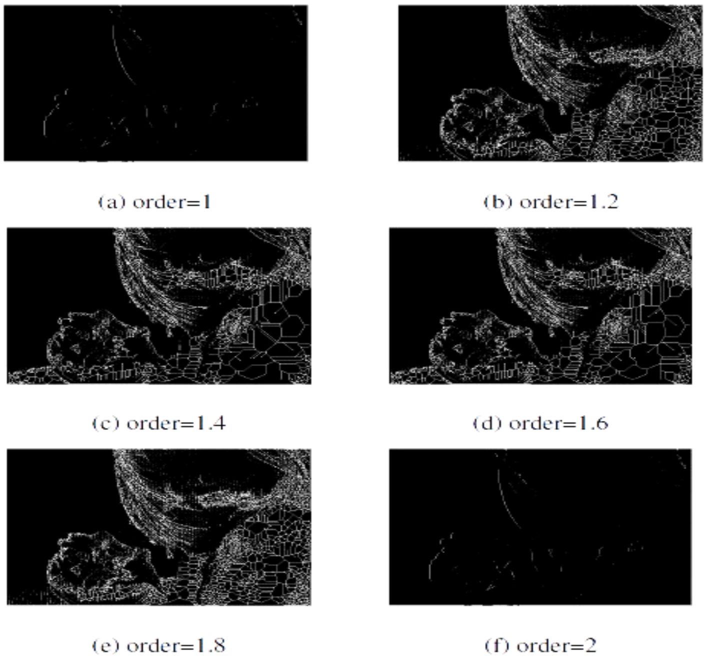
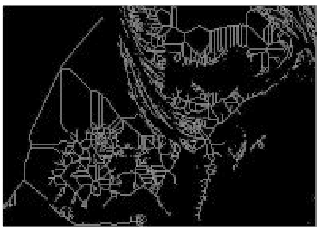


Fig.8:G-L edge detectors for non-linear.

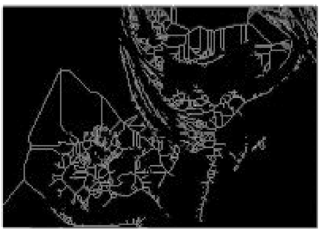
(a) order=1

(b) order=1.2

(c) order=1.4



(d) order=1.6



(e) order=1.8

(f) order=2

Fig.9: R-L edge detection fornon-linear.

From Fig.10 the FoM values for order 1.2 produces better results for both the edge detectors. But as the order is tending towards 2, R-L edge detectors showing an alternate result whereas, G-L edge detectors show better result for orders from 1.2 to 1.8.

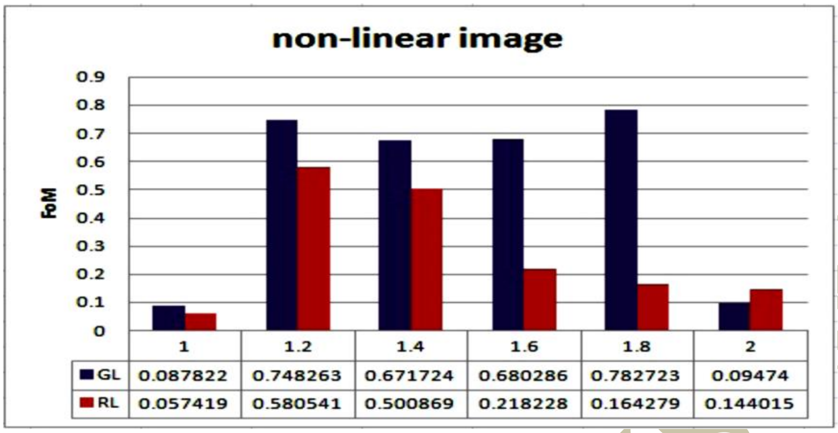


Fig.10: FoM values of G-L and R-L edge detectors

Conclusion

This paper investigates different edge detectors namely, Gradient and Laplacian edge detectors used in image processing. And compare the results with two fractional G-L and R-L edge detectors. For comparison a set of linear and non-linear image are considered in Matlab and compared with the fractional edge detectors. The different FoM are presented in the Table 1-3. From the analysis thefractional order edge detectors produces better or almost identical results when applied to linear and non-linear image as compared to classical edge detectors. Also the G-L based edge detector produces a better result.

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