Mathematical Forum ISSN: 0972-9852
Vol.28(1), 2020

STUDY OF NUMBER CONSERVING CELLULAR AUTOMATA RULES THROUGH INTEGRAL VALUE TRANSFORMATIONS

SURYAKANTA PAL[1], SUDHAKAR SAHOO[2] & BIRENDRA KUMAR NAYAK[3]

[1]*Department of Mathematics, Model Degree College,Malkangiri-764045, Odisha, India*

[2]*Institute of Mathematics and Applications, Andharua, Bhubaneswar-751029, India*

[3]*P.G. Department of Mathematics, Utkal University, Bhubaneswar-751004, India*

Email: [1]surya_tuna@yahoo.co.in, [2]sudhakar.sahoo@gmail.com, [3]bknatuu@yahoo.co.uk

Received on: 18/08/2020
Accepted on: 27/03/2021

Abstract

In this paper, the connection between two Discrete Dynamical Systems over binary strings called Cellular Automata (CA) and Integral Value Transformations (IVTs) based on their space-time diagram has been established. A new class of IVTs called number conserving IVTs (NCIVTs) having similarity with number conserving CA (NCCA) rules has been identified. The cardinality of rule space for uniform IVTs, non-uniform IVTs and non-uniform IVTs which may vary over time has been calculated for one and higher dimensions. It has been established that these NCIVTs belong to a space of exponential cardinality and their construction mechanisms are discussed. For application point of view, some interesting patterns have been generated with the application of such IVTs.

Keywords: Cellular automata, Boolean functions, space-time diagrams, number conserving cellular automata rules, number conserving integral value transformations, patterns.

2010 AMS classification: 37B15, 68Q80, 06E30, 94D10.

1. Introduction

Modeling self-reproduction with cellular automaton (pl. Cellular Automata, abbreviated as CA) was initially proposed by John von Neumann and Ulam [18**,** 11] which was further developed by John Conway and Stephen Wolfram. John Conway proposed his theory of *game of life* which has received a lot of attention among researchers. Wolfram's book "*A New Kind of Science*"[19] has attracted many researchers due to simple, scalable, robust and parallel structure of CA. CA has been used as one of the leading mathematical models to explore the complex systems such as epidemic spread model, forest fire spreading simulation, biological growth pattern etc. through its simple guiding rules. It has also applications in various field of science like physics, image processing, biology, fractal theory [5, 6], VLSI design etc. Cellular automaton is a discrete dynamical system [7, 17] consisting of four components, namely d, Q, N and f where

(i) $d = 1, 2, \ldots$ is the dimension of cellular space.

(ii) $Q = \{0, 1, 2, ..., q-1\}$ is a finite set of states associated with the cells.

(iii) N is the set of all d-dimensional neighborhood vector of size k.

(iv) $f : Q^k \to Q$ is a local update rule (or transition function) which is a mapping from set of all possible neighborhoods to set of states.

To explore the novelty of CA, Wolfram [19] introduced a very simple structure of one-dimensional, two-state and three neighborhood dependency CA called elementary CA (ECA) for which d =1, Q = {0, 1} and r =1. He used the nearest neighborhood for any cell which consists of the cell itself and one cell on the either side of the cell. So local transition function $f : Q^3 \to Q$ provides 8 neighborhood templates with N = {000, 001, 010, 011, 100, 101, 110, 111} and hence there are 2^8= 256 CA rules. The formula $c_i(t+1) = f\ (c_{i-1}(t),\ c_i(t),\ c_{i+1}(t))$ provides the next state of a cell at time step $(t+1)$ where $c_i(t)$ denotes the state of the i-th cell at time t. Wolfram introduced the naming scheme for each elementary CA rule where the rule number is obtained from the decimal conversion of the binary string $f(111)f(110)f(101)f(100)\ f(011)\ f(010)\ f(001)\ f(000)$ obtained due to assignment of the states to 8 neighborhoods. We have used one dimensional two-state CA with radius 1 under periodic boundary condition. CA rules are displayed as a lookup table listing all possible neighborhoods with a state as shown in table-1.

Different types of CA can be constructed by changing one or more parameters of CA viz., number of states, size of neighborhoods, boundary conditions, transition rules and dimension. A large variety of CA has been proposed by different researchers which are used as mathematical models to design different complex systems [20]. Some of them are Uniform CA, Non-uniform CA, Asynchronous CA, Linear CA, Non-linear CA, Reversible CA, Additive CA, Probabilistic CA etc. [3, 4].

Table 1: Shows look-up table for three elementary CA rules 184, 170 and 204 [19].

Neighborhoods	111	110	101	100	011	010	001	000	CA rules
Next states	1	0	1	1	1	0	0	0	184
Next states	1	0	1	0	1	0	1	0	170
Next states	1	1	0	0	1	1	0	0	204

In this paper, we concentrate about three types of CA, namely uniform CA, non-uniform CA and non-uniform CA varying over time *t*. In case of uniform CA, a particular rule is applied to each cell simultaneously whereas in non-uniform CA, different rules are applied at different cells. The cardinality of the rule space for different types is shown below.

(a) Number of uniform CA rules = $\left((\text{State})^{\text{State}}\right)^{\text{Neighbourhood}}$.

(b) Number of non-uniform CA rules = $\left(\left((State)^{\text{State}}\right)^{\text{Neighbourhood}}\right)^{\text{length of CA}}$.

(c) Number of non-uniform CA rules changing over different time steps throughout the evolution

$= \left(\left(\text{State}^{\text{State}^{\text{Neighborhood}}}\right)^{\text{length of the CA}}\right)^{\text{time}}$.

In literature, one finds many kinds of automaton with interesting properties out of which Integral Value Transformations (IVTs) [9] is a type of automaton denoted by $\text{IVT}_j^{p,k}$ with three components, namely,

(i) p = base of the number system.
(ii) k = number of inputs of non-negative integers called dimension.
(iii) j = rule number as per Wolfram naming convention.

An exertion has been made to urge some basic mathematical structures of IVTs like monoid, commutative ring, vector space etc. The dynamics of these IVTs over the set of natural numbers in terms of dynamical system and through topological dynamics have already been studied [8]. But some interesting properties like formation using IVTs etc. have not been explored till now. So some research work in conservation of energy, type of IVTs, equivalence with other automata, pattern this direction has been cited to enhance the beauty of IVTs.

The organization of the paper is as follows. In section-2, the drawback of CA along with basic definition of IVTs has been discussed. In section-3, the connection between two mathematical models, CA and IVTs has been established based on their space-time diagrams. In section-4, different types of IVTs like uniform IVTs, non-uniform IVTs and non-uniform IVTs varying over time have been defined. In section-5, conservation properties of IVTs have been proposed and an enumeration of number conserving IVTs is provided. In section-6, some beautiful patterns due to CA and IVTs have been generated. In section-7, some future research work has been provided.

2. Drawbacks of CA and Concept of IVTs

2.1 Drawbacks of CA

Though CA are efficient computing machines with the capability of systematic investigation of complex phenomena but it is not easy to design CA model which governs the evolution of the dynamical system. Since the local interaction is limited to a small distance so it is time consuming and not appropriate for designing a model requiring an interaction at a large distance. While implementing CA, synchronous updating of cells takes place in two phases as follows.

(1) Computation of new states for each cell based on the neighbourhoods using transition function and preservation of these new states in a temporary storage.
(2) Updating of these new states in respective cell simultaneously.

On the contrary, in case of IVTs, updating of cells does not require any overlapping neighbourhood interaction or any boundary conditions. New states can be calculated in a single phase in a sequential manner without the help of temporary storage. The synchronous approach in case of CA assumes the presence of a global clock for simultaneous updating which is practically irrelevant in real life applications. There are many situations like design of parallel carry save adder circuit, carry value transformation, in process of mutation and crossover in genetic algorithm, domain calculus in relational database management system, bitwise operators, image fusion etc. where the importance of IVTs can be realised [4, 5, 6, 12].

2.2 Concept of IVTs

Integral value transformations (IVTs) [8] are a class of discrete transformations defined on a discrete domain N_0^k where $\mathbb{N}_0$ is the set of all natural numbers including 0.

Let $(n_1, n_2, \ldots, n_k) \in \mathbb{N}_0^k$.

Then $n_1 = (a_{1n}a_{1,n-1}\ \ldots\ a_{11})_\mathrm{p}$, $n_2 = (a_{2n}a_{2,n-1}\ \ldots\ a_{21})_\mathrm{p}$, …, $n_k = (a_{kn}a_{k,n-1}\ \ldots\ a_{k1})_\mathrm{p}$.

A k-dimensional IVT in p-adic number system is a mapping $\mathrm{IVT}_j^{p,k} : \mathbb{N}_0^k \to \mathbb{N}_0$ defined as

$$\mathrm{IVT}_j^{p,k}(n_1, n_2, \ldots, n_\mathrm{k}) = (f_\mathrm{j}(a_{1n}a_{2n} \ldots a_{kn})\, f_\mathrm{j}(a_{1,n-1}a_{2,n-1} \ldots a_{k,n-1})\ \ldots\ f_\mathrm{j}(a_{11}a_{21}\ \ldots a_{k1}))_p = q.$$

where q is the decimal conversion from p-adic number, f_j is a function defined from $\{0, 1, \ldots, p\text{-}1\}^k$ to $\{0, 1, 2, \ldots, p\text{-}1\}$ and $j = 0, 1, 2, \ldots, p^{p^k}\text{-}1$.

Example 1: Let $n_1 = 220$, $j = 1$ and $p = 2$.Then $\mathrm{IVT}_1^{2,1}(220) = f_1(11011100) = (f_1(1)$ $f_1(1)f_1(0)f_1(1)\ f_1(1)f_1(1)f_1(0)f_1(0))_2 = (00100011)_2 = 35$. Thus $\mathrm{IVT}_1^{2,1}$ is a monotonically decreasing IVT converting every even number into an odd number and conversely.

Example 2: If $p = 3$ and $k = 1$, then the table-2 shows functions f_{14} and f_{19}.

Table 2: Shows functions f_{14} and f_{19}

Variables	f_{14}	f_{19}
0	2	1
1	1	0
2	1	2

$$\mathrm{IVT}_{14}^{3,1}(34) = \mathrm{IVT}_{14}^{3,1}(1021) = (f_{14}(1)\, f_{14}(0)\, f_{14}(2)\, f_{14}(1))_3 = (1211)_3 = 49.$$

$$\mathrm{IVT}_{19}^{3,1}(34) = \mathrm{IVT}_{19}^{3,1}(1021) = \left(f_{19}(1)\, f_{19}(0)\, f_{19}(2)\, f_{19}(1)\right)_3 = (0120)_3 = 15.$$

In case of binary strings, IVTs have much similarity with Bitwise operations but in general, it has lots of other variations in its architecture and topology. In this paper, we have taken $p = 2$.

3. Types of IVTs

To study conservation property of IVTs and to maintain the number of inputs same as number of outputs, the co-domain of IVTs can be taken as $\mathbb{N}_0^k$. Thus in this section, different types of IVTs are broadly classified into two types as $\mathrm{IVT}_{\#}^{2,k}:\mathbb{N}_0^k \to \mathbb{N}_0$ and $\mathrm{IVT}_{\#}^{2,k}:\mathbb{N}_0^k \to \mathbb{N}_0^k$.

3.1 Types of $\mathrm{IVT}_{\#}^{2,k}:\mathbb{N}_0^k \to \mathbb{N}_0$

(a) Uniform IVTs:

A uniform k-input IVT is a mapping $\mathrm{IVT}_i^{2,k}:\mathbb{N}_0^k \to \mathbb{N}_0$ defined as

$$\mathrm{IVT}_i^{2,k}(n_1,n_2,\ldots,n_k) = (f_i(a_{1n}a_{2n}\ \ldots\ a_{kn})\, f_i(a_{1,\,n-1}a_{2,\,n-1}\ \ldots\ a_{k,\,n-1})\ \ldots\ f_i(a_{11}a_{21}\ \ldots\ a_{k1}))_2 = q.$$

where q is the decimal conversion from the binary system, f_i is a Boolean function of k variables defined from $\{0, 1\}^k$ to $\{0, 1\}$ and $i = 0, 1, 2, \ldots, 2^{2^k}-1$.

Example 3:

Let $n_1 = 122,\ n_2 = 108,\ n_3 = 43.$

Then $n_1 = (1111010)_2,\ n_2 = (1101100)_2,\ n_3 = (101011)_2 = (0101011)_2.$

$$\mathrm{IVT}_{226}^{2,3}(122,108,43) = \mathrm{IVT}_{226}^{2,3}(1111010,\ 1101100,\ 0101011)$$
$$= (f_{226}(110)\, f_{226}(111)\, f_{206}(100)\, f_{206}(111)\, f_{206}(010)\, f_{206}(101)\, f_{206}(001))_2$$
$$= (1101011)_2 = 107.$$

(b) Non-uniform IVTs:

Like non-uniform CA, application of different IVTs at different position of the strings leads to a non-uniform IVT and it is denoted as $\mathrm{IVT}_{<i_1,i_2,\ \ldots,\ i_n>}^{2,k}(n_1,n_2,\ \ldots,\ n_k)$. Thus a non-uniform IVT over k nonnegative integers is a mapping $\mathrm{IVT}_{<i_1,i_2,\ \ldots,\ i_n>}^{2,k}:\ \mathbb{N}_0^k \to \mathbb{N}_0$ defined as $\mathrm{IVT}_{<i_1,i_2,\ldots,i_n>}(n_1,n_2,\ldots,n_k) =$

$$(f_{i_1}(a_{1n}a_{2n}\ \ldots\ a_{kn})\, f_{i_2}(a_{1,n-1}a_{2,n-1}\ \ldots\ a_{k,n-1})\ \ldots\ f_{i_n}(a_{11}a_{21}\ \ldots\ a_{k1}))_2 = q$$

where f_{i_j} is a Boolean function of k variables, each integer n_i is represented as an n-bit string.

Example 4:

$$\mathrm{IVT}_{<184,\ 204,\ 226,\ 232,\ 240,170,\ 184>}^{2,3}(120,\ 108,\ 40) = \mathrm{IVT}_{<184,\ 204,\ 226,\ 232,\ 240,\ 170,\ 184>}^{2,3}(1111000,\ 1101100,\ 0101000) = (f_{184}(110)\, f_{204}(111)\, f_{226}(100)\, f_{232}(111)\, f_{240}(010)\, f_{170}(000)\, f_{184}(000))_2 = (0101000)_2 = 40.$$

(c) Non-uniform IVTs varying with respect to time *t*:

It is possible to change a non-uniform rule vector over time which is necessarily a composition function.

Let $\mathrm{IVT}_{<r_{11}, r_{12}, \ldots, r_{1n}>}, \mathrm{IVT}_{<r_{21}, r_{22}, \ldots, r_{1n}>}, \ldots, \mathrm{IVT}_{<r_{m1}, r_{m2}, \ldots, r_{mn}>}$ be the different non-uniform IVTs acting at different time steps $t = 1, 2, \ldots, m$.

This non-uniform IVTs over k nonnegative integers that vary over time t, $t \leq k$ is denoted by $\mathrm{IVT}^{2,.k}_{<r_{11}, r_{12}, \ldots, r_{1n}>, <r_{21}, r_{22}, \ldots, r_{1n}>, \ldots., <r_{t1}, r_{t2}, \ldots, r_{tn}>}(n_1, n_2, \ldots, n_k)$ and can be defined recursively as follows.

Let $f_{<r_{11}, r_{12}, \ldots, r_{1n}>}(n_1, n_2, \ldots, n_k) = q_1$, $f_{<r_{21}, r_{22}, \ldots, r_{2n}>}(n_2, \ldots, n_k, q_1) = q_2$,

$f_{<r_{31}, r_{32}, \ldots, r_{3n}>}(n_3, \ldots, n_k, q_1, q_2) = q_3$, $\ldots f_{<r_{i1}, r_{i2}, \ldots, r_{in}>}(n_i, \ldots, n_k, q_1, q_2, \ldots, q_{i-1}) = q_i$, $\ldots$

Thus for computing i-th configuration of the space-time diagram, we need its previous k configurations.

Example 5: The output of non-uniform IVT varying over time on three inputs 28, 23 and 30 is shown below.

$t = 0$:

11100 = 28
10111 = 23
11110 = 30

$t = 1$: $\mathrm{IVT}^{2,\,3}_{<170,\,226,\,180,\,185,\,234>}$

10111 = 23
11110 = 30
11110 = 30

$t = 2$: $\mathrm{IVT}^{2,\,3}_{<184,\,187,\,240,\,250,\,190>}$

11110 = 30
11110 = 30
11111 = 31

$t = 3$: $\mathrm{IVT}^{2,\,3}_{<187,\,31,\,62,\,206,\,220>}$

11110 = 30
11111 = 31
10010 = 18

In next section, k number of IVTs (uniform or non-uniform) applied to k number of non-negative integers has been defined.

3.2 Types of $\mathrm{IVT}_{\#}^{2,k}:\mathbb{N}_0^k \to \mathbb{N}_0^k$

(a) Uniform IVTs:

If $i_1, i_2, \ldots, i_k$ are k different rule numbers, then the corresponding uniform IVT is a mapping $\mathrm{IVT}^{2,k}_{i_1,\, i_2,\, \ldots,\, i_k}:\mathbb{N}_0^k \to \mathbb{N}_0^k$ defined as

$$\mathrm{IVT}^{2,k}_{i_1,i_2,\ldots,i_k}(n_1,n_2,\ldots,n_k) = (f_{i_1}(n_1,n_2,\ldots,n_k),\ f_{i_2}(n_1,n_2,\ldots,n_k),\ \ldots,\ f_{i_k}(n_1,n_2,\ldots,n_k)).$$

where f_{i_j} is a Boolean function of k variables defined from $\{0, 1\}^k$ to $\{0, 1\}$ and i_j = 0, 1, 2, …, 2^{2^k} -1.

Example 6:

$\mathrm{IVT}^{2,\,3}_{190,\ 204,\ 226}(30,\ 64,\ 85) = (\mathrm{IVT}^{2,\,3}_{190}(30,\ 64,\ 85)\mathrm{IVT}^{2,\,3}_{204}(30,\ 64,\ 85)$
$\mathrm{IVT}^{2,\,3}_{226}(30,\ 64,\ 85))$=($f_{190}$(0011110, 1000000, 1010101), f_{204}(0011110, 1000000, 1010101), f_{226}(0011110, 1000000, 1010101)) = (1011111, 1000000, 0010101) = (95, 64, 21).

(b) Non-uniform IVTs:

If $<i_{11},\, i_{12},\, \ldots,\, i_{1n}>,\ <i_{21},\, i_{22},\, \ldots,\, i_{2n}>,\ \ldots,\ <i_{k1},\, i_{k2},\, \ldots,\, i_{kn}>$ are k different rule numbers, then the corresponding non-uniform IVT is a mapping $\mathrm{IVT}^{2,\,k}_{<i_{11},\, i_{12},\, \ldots,\, i_{1n}>,\ <i_{21},\, i_{22},\, \ldots,\, i_{2n}>,\ \ldots,\ <i_{k1},\, i_{k2},\, \ldots,\, i_{kn}>}:\mathbb{N}_0^k \to \mathbb{N}_0^k$ defined as

$$\mathrm{IVT}^{2,\,k}_{<i_{11},\, i_{12},\, \ldots,\, i_{1n}>,\ <i_{21},\, i_{22},\, \ldots,\, i_{2n}>,\ \ldots,\ <i_{k1},\, i_{k2},\, \ldots,\, i_{kn}>}(n_1,n_2,\ldots,n_k) = (f_{<i_{11},\, i_{12},\, \ldots,\, i_{1n}>}(n_1,n_2,\ldots,n_k),$$
$$f_{<i_{21},\, i_{22},\, \ldots,\, i_{2n}>}(n_1,n_2,\ldots,n_k),\ \ldots,\ f_{<i_{k1},\, i_{k2},\, \ldots,\, i_{kn}>}(n_1,n_2,\ldots,n_k)) = (p_1,\, p_2,\ \ldots,\, p_k)$$

where p_i is a non-negative integer obtained in the same procedure as described above in section 3.1(b).

Example 7:

For k = 2, $\mathrm{IVT}^{2,2}_{<10,\ 12,7,5>,<4,6,10,6>}(13,14) = (f_{<10,\ 12,7,5>}(13,14),\ f_{<4,6,10,6>}(13,14))$
$= (f_{<10,\ 12,7,5>}(1101,1110),\ f_{<4,6,10,6>}(1101,1110))$
$= (f_{10}(11)\, f_{12}(11)\, f_7(01)\, f_5(10),\ \ f_4(11)\, f_6(11)\, f_{10}(01)\, f_6(10)) = (1111,0011) = (15,3).$

(c) Non-uniform IVTs varying in each time step

Let $\mathrm{IVT}^{2,\,k}_{<i_{11},\, i_{12},\ldots,\, i_{1n}>,<i_{21},\, i_{22},\ldots,\, i_{2n}>,\ \ldots,<i_{k1},\, i_{k2},\ldots,\, i_{kn}>}$, $\mathrm{IVT}^{2,\,k}_{<j_{11},\, j_{12},\ldots,\, j_{1n}>,<j_{21},\, j_{22},\ldots,\, j_{2n}>,\ \ldots,\ <j_{k1},\, j_{k2},\ \ldots,\, j_{kn}>}$, …, $\mathrm{IVT}^{2,\,k}_{<u_{11},\, u_{12},\ \ldots,\, u_{1n}>,\ <u_{21},\, u_{22},\ \ldots,\, u_{2n}>,\ \ldots,\ <u_{k1}, u_{k2},\ \ldots,\, u_{kn}>}$ be the different non-uniform IVTs acting at different time steps.

At $t = 1$, $\mathrm{IVT}^{2,\,k}_{<i_{11},\, i_{12},\, \ldots,\, i_{1n}>,\, <i_{21},\, i_{22},\, \ldots,\, i_{2n}>,\, \ldots,\, <i_{k1},\, i_{k2},\, \ldots,\, i_{kn}>}(n_1, n_2, \ldots, n_k) = (p_{11}, p_{12}, \ldots, p_{1k})$

At $t = 2$, $\mathrm{IVT}^{2,\,k}_{<j_{11},\, j_{12},\, \ldots,\, j_{1n}>,\, <j_{21},\, j_{22},\, \ldots,\, j_{2n}>,\, \ldots,\, <j_{k1},\, j_{k2},\, \ldots,\, j_{kn}>}(p_{11}, p_{12}, \ldots, p_{1k})$

$= (p_{21}, p_{22}, \ldots, p_{2k})$ and so on.

Thus for computing i-th configuration of this space-time diagram, we use k nonnegative integers obtained in its (i-1)-th configuration.

Example 8:

The output obtained from same three inputs 28, 23 and 30 is shown below.

$t = 0$:

11100 = 28
10111 = 23
11110 = 30

$t = 1$: $\mathrm{IVT}^{2,\,3}_{<170,\, 226,\, 180,\, 185,\, 234>,<184,\, 187,\, 240,\, 250,\, 190>,<187,\, 170,\, 175,\, 206,\, 220>}$

11110 = 30
11111 = 31
11111 = 31

$t = 2$: $\mathrm{IVT}^{2,\,3}_{<184,\, 190,\, 180,\, 185,\, 226>,<185,\, 220,\, 204,\, 240,\, 231>,<185,\, 175,\, 170,\, 204,\, 204>}$

11110 = 30
11110 = 30
11111 = 31

4. Connection between CA and IVTs

To study property of CA rules, study of IVTs might be helpful as both share a common paradigm of Boolean functions. In this section, a meaningful relationship between IVTs and CA rules [14] has been cited which claims that the different issues in the field of cellular automata can be addressed through IVTs.

(a) Connection between uniform CA and uniform IVTs:

Let $a_0a_1 \ldots a_{n-1}a_n$ be any configuration of CA.

Let f_j be the local function of CA rule-j and G be its global function.

Let $a = (a_0a_1 \ldots a_{n-1}a_n)_2$, $a' = (a_na_0 \ldots a_{n-2}a_{n-1})_2$ and $a'' = (a_1a_2 \ldots a_na_0)_2$.

$\mathrm{G}(a) = \mathrm{G}(a_0a_1 \ldots a_{n-1}a_n) = f_j(a_na_0a_1)\, f_j(a_0a_1a_2) \ldots f_j(a_{n-1}a_na_0)$

Now $(f_j(a_na_0a_1)\, f_j(a_0a_1a_2) \ldots f_j(a_{n-1}a_na_0))_2 = \mathrm{IVT}_j^{2,\,3}(a_na_0a_1 \ldots a_{n-2}a_{n-1},$

$a_0a_1a_2 \ldots a_{n-1}a_n, \ldots, a_1a_2 \ldots a_{n-1}a_na_0) = \mathrm{IVT}_j^{2,\,3}(a', a, a'')$.

Thus the decimal value of the output configuration of an elementary CA rule-j acting on a configuration $a_0a_1 \ldots a_{n-1}a_n$ is same as $\mathrm{IVT}_j^{2,3}(a^/, a, a^{//})$ where $a^/ = (a_na_0 \ldots a_{n-2}a_{n-1})_2$, $a = (a_0a_1 \ldots a_{n-1}a_n)_2$ and $a^{//} = (a_1a_2 \ldots a_na_0)_2$.

Example 9:
Let a = 349 = $(101011101)_2$. Then $a^/$ =110101110 and $a^{//}$=010111011
$f_j(a)=(f_j(110)\, f_j(101)\, f_j(010)\, f_j(101)\, f_j(011)\, f_j(111)\, f_j(110)\, f_j(101)\, f_j(011))_2$

$$= \mathrm{IVT}_j^{2,3}(110101110,\ 101011101,\ \ldots,\ 010111011)$$

$$= \mathrm{IVT}_j^{2,3}(430,349,187) = \mathrm{IVT}_j^{2,3}(a^/, a, a^{//})$$

<u>Values of $a^/$ and $a^{//}$ in terms of a:</u>

$$a = a_0 \times 2^n + a_1 \times 2^{n-1} + \ldots + a_{n-1} \times 2^1 + a_n \times 2^0.$$

$$a^/ = a_n \times 2^n + \mathrm{a}_0 \times 2^{n-1} + a_1 \times 2^{n-2} + \ldots + a_{n-2} \times 2^1 + a_{n-1} \times 2^0.$$

$$= \frac{1}{2}\left[a_n \times 2^{n+1} + \mathrm{a}_0 \times 2^n + a_1 \times 2^{n-1} + \ldots + a_{n-2} \times 2^2 + a_{n-1} \times 2^1\right] = \frac{1}{2}\left[a_n \times 2^{n+1} + a - a_n\right]$$

$$= \frac{1}{2}\left[a + a_n(2^{n+1} - 1)\right]$$

$$= \begin{cases} \dfrac{a}{2} & \text{if } a_n = 0 \\ \dfrac{a-1}{2} + 2^{\mathrm{n}} & \text{if } a_n = 1 \end{cases} \Rightarrow a^/ = \begin{cases} \dfrac{a}{2} & \text{if } a \text{ is even} \\ \dfrac{a-1}{2} + 2^n & \text{if } a \text{ is odd} \end{cases}$$

where n = length of string = $l(a)$.

$$a^{//} = a_1 \times 2^n + a_2 \times 2^{n-1} + \ldots + a_n \times 2^1 + a_0 \times 2^0$$

$$= 2\left(a_1 \times 2^{n-1} + a_2 \times 2^{n-2} + \ldots + a_n \times 2^0 + a_0 \times \frac{1}{2}\right)$$

$$= 2\left(a - a_0 \times 2^n + \frac{a_0}{2}\right) = 2\left[a + a_0\left(\frac{1}{2} - 2^n\right)\right] = \begin{cases} 2a & \text{if } a_0 = 0 \\ 2a + 1 - 2^{n+1} & \text{if } a_0 = 1 \end{cases}$$

Thus an elementary CA rule-j acting on a configuration $a_0a_1a_2 \ldots a_{n-1}a_n$ generates same output as an $\mathrm{IVT}_j^{2,3}(a^/, a, a^{//})$ where $a^/ = \frac{1}{2}\left[a + a_n(2^{n+1} - 1)\right]$ and $a^{//} =$ $2\left[a + a_0\left(1/2 - 2^n\right)\right]$ with a_0 and a_n are most and least significant digits in binary expansion of a.

(b) Connection between non-uniform CA and non-uniform IVTs:

Let $\mathrm{F}_{<i_0,\, i_1,\, i_2,\, \ldots,\, i_n>}$ be the global function of a non-uniform CA and f_j be the local function of CA rule-j.

$\mathrm{F}_{<i_0,\, i_1,\, i_2,\, \ldots,\, i_n>}(a) = f_{i_0}(a_n a_0 a_1)\, f_{i_1}(a_0 a_1 a_2) \ldots\ f_{i_n}(a_{n-1} a_n a_0)$

Its decimal value is $(f_{i_0}(a_n a_0 a_1)\, f_{i_1}(a_0 a_1 a_2) \ldots\ f_{i_n}(a_{n-1} a_n a_0))_2$

$= \mathrm{IVT}^{2,\,3}_{<i_0,\, i_1,\, \ldots,\, i_n>}(\, a_n a_0 a_1 a_2 \ldots a_{n-2} a_{n-1},\ a_0 a_1 a_2 \ldots a_{n-1} a_n,\ \ldots,\ a_1 a_2 \ldots a_{n-1} a_n a_0\,)$

$= \mathrm{IVT}^{2,\,3}_{<i_0,\, i_1,\, \ldots,\, i_n>}(a^{/},\ a,\ a^{//})$

In general, dimension k of IVT = size of the neighborhood used in the corresponding CA and the states associated with each configuration of a CA can be used to form k nonnegative integers which are used as k inputs of the corresponding IVT.

5. Number Conserving IVTs (NCIVTs):

The concept of number conserving IVTs has a similarity with a particular class of CA called number conserving cellular automata (NCCA) [1] which have attracted many researchers to study their dynamical behaviors such as decidability, reversibility, stability, periodicity etc. One of the additive invariant in CA is sum of the values of all cells in a finite lattice over the periodic boundary condition remains constant throughout the evolution. The CA rules obeying such invariants are called number conserving CA rules [2, 10]. Out of 256-elementary uniform CA rules, five rules, namely Rule-204, Rule-170, Rule-184, Rule-226 and Rule-240 are number conserving CA under periodic boundary condition. The similarity between Number Conserving IVTs (NCIVTs) and Number Conserving Cellular Automata (NCCA) rules has been explored in next section.

To discuss the conservation property of IVTs, we have used the principle that the density of 1s in input arguments is same as density of 1s in output arguments. So we should have the condition that number of inputs = number of outputs which motivate us to study $\mathrm{IVT}^{2,\,k}_{\#} : \mathbb{N}_0^k \to \mathbb{N}_0^k$ for conservation.

(a) NCIVTs in one dimension (k = 1):

(i) 1D Uniform NCIVTs:

A one-dimensional $\mathrm{IVT}^{2,\,1}_j : \mathbb{N}_0 \to \mathbb{N}_0$ is number conserving if $\mathrm{IVT}^{2,\,1}_j(x) = q$ implies that $w(x) = w(q)$. The only trivial one-dimensional number conserving IVT is $\mathrm{IVT}^{2,\,1}_2$ which conserves both weight and decimal value as it is an identity mapping.

Table 3: Truth table of all Boolean functions in one variable

x_1	f_0	f_1	f_2	f_3
0	0	1	0	1
1	0	0	1	1

Result 5.1: There exists no nontrivial one-dimensional uniform number conserving IVTs.

Proof: Suppose there exists a one-dimensional number conserving IVT, say, $\mathrm{IVT}_i^{2,1}$.

So $\mathrm{IVT}_i^{2,1}(00 \ldots 00) = 0 = (00 \ldots 00)_2 \Rightarrow \mathrm{IVT}_i^{2,1} = \mathrm{IVT}_0^{2,1}$ or $\mathrm{IVT}_2^{2,1}$.

Similarly, $\mathrm{IVT}_i^{2,1}(11 \ldots 11) = (11 \ldots 11)_2 \Rightarrow \mathrm{IVT}_i^{2,1} = \mathrm{IVT}_1^{2,1}$ or $\mathrm{IVT}_3^{2,1}$.

Since there is no common IVTs so there exists no one-dimensional number conserving IVTs.

(ii) 1D Non-Uniform NCIVT:

Application of four different one dimensional IVTs at different position of the string leads to a non-uniform IVT rule vector. Number of 1D non-uniform IVT rules is 4^n where n is the length of the string. If we apply different non-uniform IVT rule vectors of size n for t times, then we get a rule matrix [13] of order $t \times n$. Number of such rule matrices is $(4^n)^t$.

Result 5.2: There exists no nontrivial one-dimensional non-uniform number conserving IVTs.

Proof: Suppose there exists a one-dimensional non-uniform number conserving IVTs, say, $\mathrm{IVT}^{2,k}_{<r_{i1}, r_{i2}, \ldots, r_{in}>}$.

$\mathrm{IVT}^{2,k}_{<r_{i1}, r_{i2}, \ldots, r_{in}>}(00\ldots00) = 0 = (00\ldots00)_2$. So each $r_{ij} = 0$ or 2.

$\mathrm{IVT}^{2,k}_{<r_{i1}, r_{i2}, \ldots, r_{in}>}(11 \ldots 11) = (11\ldots 11)_2$. So each $r_{ij} = 1$ or 3.

There is no common rule vector and hence no IVTs are possible. So there exists no one-dimensional non-uniform NCIVT.

It may be noted that the architecture of both CA and IVTs in one dimension are same.

Our next focus will be to search the number conserving IVTs in 2D and 3D.

(b) NCIVTs in higher dimensions (*k* > 1)

(i) 2D Uniform NCIVT:

Let $a_1, a_2, \ldots, a_m \in \mathbb{N}_0$. Weight of m nonnegative integers is denoted by $w(a_1, a_2, \ldots, a_m)$ and is defined as $w(a_1, a_2, \ldots, a_m) = w(a_1) + w(a_2) + \ldots + w(a_m)$ where $w(a_1), w(a_2), \ldots, w(a_m)$ are the number of 1s in the binary representations of $a_1, a_2, \ldots, a_m$.

$\mathrm{IVT}_{i,j}^{2,2}$ in two dimension is defined as $\mathrm{IVT}_{i,j}^{2,2}(a,b) = (f_i(a,b), f_j(a,b))$ for all $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$\mathrm{IVT}_{i,j}^{2,2}$ is number conserving if $w(a, b) = w(f_i(a,b), f_j(a,b))$.

Theorem 1: The two dimensional uniform number conserving rules are $\mathrm{IVT}_{10,12}^{2,2}$, $\mathrm{IVT}_{12,10}^{2,2}$, $\mathrm{IVT}_{8,14}^{2,2}$ and $\mathrm{IVT}_{14,8}^{2,2}$.

Proof: Let $n_1, n_2 \in \mathbb{N}_0$. Let $n_1 = (a_n a_{n-1} \ldots a_1)_2$, $n_2 = (b_n b_{n-1} \ldots b_1)_2$.

$\mathrm{IVT}_{i,j}^{2,2}(n_1,n_2) = (f_i(n_1,n_2), f_j(n_1,n_2)) = (f_i(a_n b_n) f_i(a_{n-1}b_{n-1}) \ldots f_i(a_0 b_0), f_j(a_n b_n)$
$f_j(a_{n-1}b_{n-1}) \ldots f_j(a_1 b_1))$

Suppose there exists a 2D uniform number conserving rule $\mathrm{IVT}_{i,j}^{2,2}$ for some values i and j.

$\mathrm{IVT}_{i,j}^{2,2}(0,0) = (f_i(0,0), f_j(0,0)) = (f_i(00) f_i(00) \ldots f_i(00), f_j(00) f_j(00) \ldots f_j(00))$

Since $f_{i,j}$ is number conserving so $(f_i(00) f_i(00) \ldots f_i(00), f_j(00) f_j(00) \ldots f_j(00)) = (00\ldots0, 00\ldots0)$.

So $f_i(00) = 0$ and $f_j(00) = 0$……………………………………………………………(1)

Similarly, $f_i(11) = 1$ and $f_j(11) = 1$…………………………………………………..(2)

If $a_k = (b_k)^c$ for some k, then for number conserving, we must have $f_i(a_k b_k) = [f_j(a_k b_k)]^c$.

[It is because f_i and f_j act directly on $a_k b_k$ without having relationship with other bits.]

If $a_k = 0$, then $b_k = 1$. So either [$f_i(a_k b_k) = 0$ and $f_j(a_k b_k) = 1$] or [$f_i(a_k b_k) = 1$ and $f_j(a_k b_k) = 0$].

$\Rightarrow$ [$f_i(01) = 0$ and $f_j(01) = 1$] or [$f_i(01) = 1$ and $f_j(01) = 0$]……………………(3)

If $a_k = 1$, then $b_k = 0$. So either [$f_i(a_k b_k) = 1$ and $f_j(a_k b_k) = 0$] or [$f_i(a_k b_k) = 0$ and $f_j(a_k b_k) = 1$]

$\Rightarrow$ [$f_i(10) = 0$ and $f_j(10) = 1$] or [$f_i(10) = 1$ and $f_j(10) = 0$]……………………(4)

Combining all possibilities, we have four possibilities.

One of them is $f_i(00) = 0$ and $f_j(00) = 0$, $f_i(01) = 0$ and $f_j(01) = 1$, $f_i(10) = 0$ and $f_j(10) = 1$, $f_i(11) = 1$ and $f_j(11) = 1$.

i.e. $f_i(00) = 0$, $f_i(01) = 0$, $f_i(10) = 0$ and $f_i(11) = 1$. Again, $f_j(00) = 0$, $f_j(01) = 1$, $f_j(10) = 1$ and $f_j(11) = 1 \Rightarrow i = (1000)_2 = 8$ and $j = (1110)_2 = 14$.

Similarly, considering other three possibilities, we get

(a) $i = 14$ and $j = 8$ (b) $i = 10$ and $j = 12$ (c) $i = 12$ and $j = 10$.

So the two dimensional uniform NCIVTs are $\mathrm{IVT}_{10,12}^{2,2}$, $\mathrm{IVT}_{12,10}^{2,2}$, $\mathrm{IVT}_{8,14}^{2,2}$ and $\mathrm{IVT}_{14,8}^{2,2}$ which can also be constructed in the table-4 and shown in table-5.

Note that $f_{i,j}(01) = \{0, 1\}$ means $f_i(01) = 0$ and $f_j(01) = 1$.

Number of all possible NCIVTs is $1\times2\times2\times1=4$.

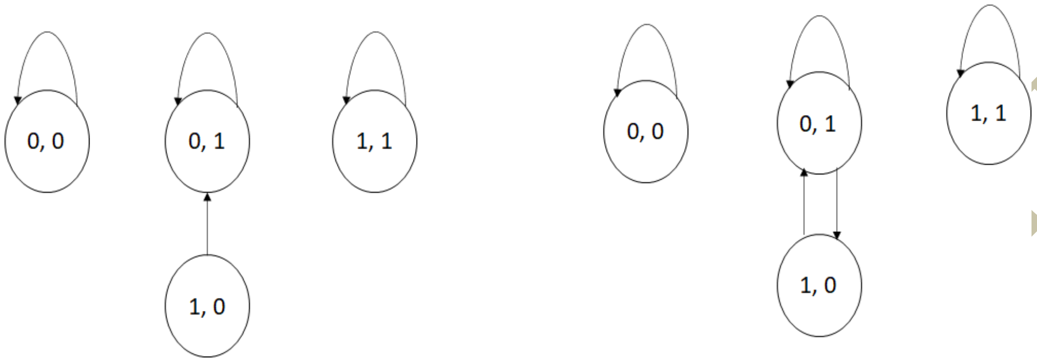
Table 4: Truth table of all possible NCIVTs

x_1	x_2	$f_{i..j}$
0	0	{0, 0}
0	1	{0, 1}, {1, 0}
1	0	{1,0}, {0, 1}
1	1	{1, 1}

Table 5: Truth table of all 2D uniform NCIVTs

x_1	x_2	$f_{10,12}$	$f_{12,10}$	$f_{8,14}$	$f_{14,8}$
0	0	{0,0}	{0,0}	{0,0}	{0,0}
0	1	{1,0}	{0,1}	{0,1}	{1,0}
1	0	{0,1}	{1,0}	{0,1}	{1,0}
1	1	{1,1}	{1,1}	{1,1}	{1,1}

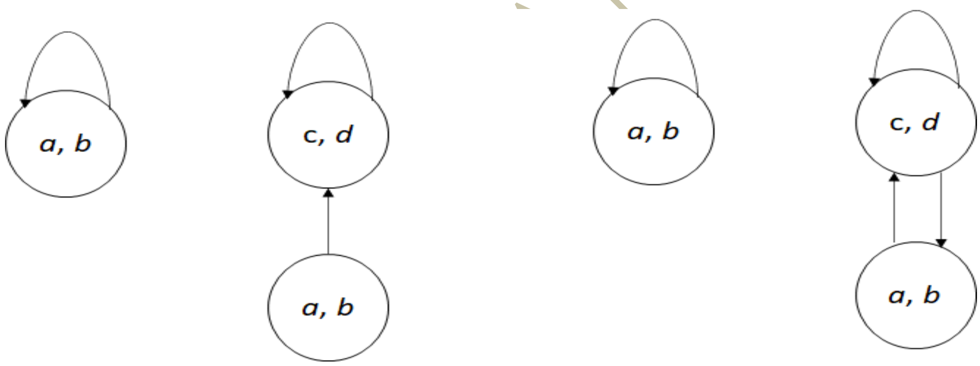
STDs of basis of both $f_{8,14}$ and $f_{10,12}$ are shown in Fig. 1.



(a) **(b)**

Fig.1: The STDs of $f_{8,14}$ and $f_{10,12}$

STDs of two nonnegative integers *a* and*b* are shown in Fig. 2.



(a) **(b)**

Fig. 2: The STDs of two NCIVTs (a) $\mathrm{IVT}_{8,\,14}^{2,\,2}$ and (b) $\mathrm{IVT}_{10,\,12}^{2,\,2}$

Example 10:

$\mathrm{IVT}_{10,\,12}^{2,\,2}(140,\,182) =$

$(f_{10}(140,182),\, f_{12}(140,182)) = (f_{10}(10001100,10110101),\, f_{12}(10001100,10110101))$

$= (f_{10}(11)\, f_{10}(00)\, f_{10}(01)\, f_{10}(01)\, f_{10}(10)\, f_{10}(11)\, f_{10}(00)\, f_{10}(01),$

$f_{12}(11)\, f_{12}(00)\, f_{12}(01)\, f_{12}(01)\, f_{12}(10)\, f_{12}(11)\, f_{12}(00)\, f_{12}(01))$

$= (10110101, 10001100) = (182, 140).$

We observe that input sum = output sum.

(ii) 2D Non-Uniform NCIVTs:

Let $n_1 = (a_n a_{n-1} \ldots a_1 a_0)_2$ and $n_2 = (b_n b_{n-1} \ldots b_1 b_0)_2$. A non-uniform IVT over two nonnegative integers is given by

$$\mathrm{IVT}^{2,2}_{<i_{11}, i_{12}, \ldots, i_{1n}>, <i_{21}, i_{22}, \ldots, i_{2n}>}(n_1, n_2) = (f_{<i_{11}, i_{12}, \ldots, i_{1n}>}(n_1, n_2), f_{<i_{21}, i_{22}, \ldots, i_{2n}>}(n_1, n_2))$$
$$= (f_{<i_{11}, i_{12}, \ldots, i_{1n}>}(a_n a_{n-1} \ldots a_1, b_n b_{n-1} \ldots b_1), f_{<i_{21}, i_{22}, \ldots, i_{2n}>}(a_n a_{n-1} \ldots a_1, b_n b_{n-1} \ldots b_1))$$
$$= (f_{i_{11}}(a_n b_n) f_{i_{12}}(a_{n-1} b_{n-1}) \ldots f_{i_{1n}}(a_1 b_1), f_{i_{21}}(a_n b_n) f_{i_{22}}(a_{n-1} b_{n-1}) \ldots f_{i_{2n}}(a_1 b_1))$$
$$= (p_1, p_2).$$

A non-uniform $\mathrm{IVT}^{2,2}_{<i_{11}, i_{12}, \ldots, i_{1n}>, <i_{21}, i_{22}, \ldots, i_{2n}>}$ is number conserving if $w(n_1, n_2) = w(p_1, p_2)$.

Theorem 2: The two dimensional non-uniform number conserving IVTs are $\mathrm{IVT}^{2,2}_{<10, 10, \ldots, 10>, <12, 12, \ldots, 12>}$, $\mathrm{IVT}^{2,2}_{<12, 12, \ldots, 12>, <10, 10, \ldots, 10>}$, $\mathrm{IVT}^{2,2}_{<8, 8, \ldots, 8>, <14, 14, \ldots, 14>}$, $\mathrm{IVT}^{2,2}_{<14, 8, \ldots, 14>, <8, 14, \ldots, 8>}$ and so on.

Proof: It is similar to the proof of the last theorem.

Number of such two dimensional non-uniform number conserving rules is 4^n.

Number of such two dimensional non-uniform number conserving rules varying over time t is $(4^n)^t$.

Example 11:

$$\mathrm{IVT}^{2,2}_{<8, 14, 8, 8, 14, 14>, <14, 8, 14, 14, 8, 8>}(43, 50)$$
$$= \mathrm{IVT}^{2,2}_{<8, 14, 8, 8, 14, 14>, <14, 8, 14, 14, 8, 8>}(101011, 110010)$$
$$= (f_8(11) f_{14}(01) f_8(10) f_8(00) f_{14}(11) f_{14}(10), f_{14}(11) f_8(01) f_{14}(10) f_{14}(00) f_8(11) f_8(10))$$
$$= (110011, 101010) = (51, 42).$$

Example 12: The output of non-uniform NCIVT $f_{<10, 12, 10, 12, 10, 12, 10, 12>, <12, 10, 12, 10, 12, 10, 12, 10>}$ over two inputs 147 and 197 is shown below.

$t = 1$:

10010011=147

11000101=197

$t = 2$:

10010001 = 145

11000111 = 199

$t = 3$:
10010001=145
11000111=199
At all-time steps, the sum of two integers remains same and this is also same as the sum of two initial inputs. In this example, 147 + 197 = 145 + 199 = 344.
If we plot these types of points in 2D, then they remain in a straight line $x + y = c$ as shown in Fig. 3.

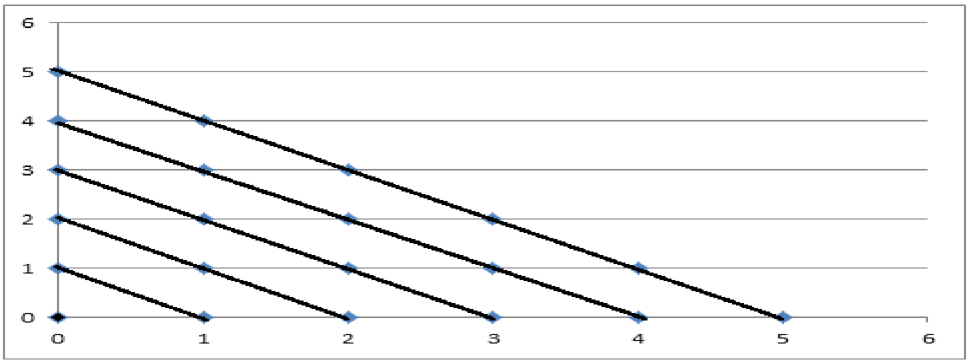


Fig.3: Lines are drawn between $(0, z)$ and $(z, 0)$ for all non-negative integer z.

(iii) 3D Uniform NCIVT:

$\mathrm{IVT}^{2,3}_{i,j,k}$ is number conserving if for all $(n_1, n_2, n_3) \in \mathbb{N}_0^3$, $\mathrm{IVT}^{2,3}_{j}(n_1, n_2, n_3) = (p_1, p_2, p_3)$ implies that $w(p_1, p_2, p_3) = w(n_1, n_2, n_3)$.

Table 6: Truth table of all possible 3D NCIVTs

x_1	x_2	x_3	$f_{i..j,k}$
0	0	0	{0, 0, 0}
0	0	1	{0,0,1}, {0,1, 0}, {1, 0, 0}
0	1	0	{0, 1,0}, {0,0, 1},{1, 0, 0}
0	1	1	{0, 1, 1}, {1, 0, 1}, {1, 1, 0}
1	0	0	{1, 0, 0}, {0, 0, 1}, {0, 1, 0}
1	0	1	{1, 0, 1}, {0, 1, 1}, {1, 1, 0}
1	1	0	{1, 1, 0}, {1, 0, 1}, {0, 1, 1}
1	1	1	{1,1,1}

Note that $f_{i,j,k}(010) = \{1, 0, 0\}$ means $f_i(010) = 1$, $f_j(010) = 0$ and $f_k(010) = 0$.
Number of all possible NCIVTs is $1\times3\times3\times3\times3\times\times3\times3\times1 = 3^6 = 729$ which can be constructed from table-6.

Theorem 3: The cardinality of the set of three dimensional uniform NCIVTs is 729. Some of them are $IVT^{2,3}_{128,232,254}$, $IVT^{2,3}_{138,228,248}$, $IVT^{2,3}_{152,226,236}$, $IVT^{2,3}_{160,218,236}$, $IVT^{2,3}_{160,200,254}$ etc. So number of 3-D non-uniform number conserving rules is $(729)^n$ and number of 3-D non-uniform number conserving rules varying over time t is $(729^n)^t$.

In general, the number of m-dimensional uniform NCIVTs is $N = C(m,0)^{C(m,0)} \times C(m,1)^{C(m,1)} \times \ldots \times C(m,m)^{C(m,m)}$. The number of m-dimensional non-uniform NCIVTs is N^n where $n = \max\{l(n_1), l(n_2), \ldots, l(n_k)\}$ whereas number of such 3-D non-uniform number conserving rules varying over time t is $(N^n)^t$.

6. Pattern formation using CA and IVTs

The space-time diagram of different types of one dimensional CA rules and their characteristics can be visualized through patterns [15,16]. The following patterns are obtained on application of elementary CA rules starting from two initial rows of configurations. In this case, every cell has six possible neighborhoods like cell, left (L), right (R), top (T), left corner (LC) and right corner (RC) from which we have considered only three. If $a = (a_0a_1\ldots a_{n-1}a_n)_2$ and $b = (b_0b_1\ldots b_{n-1}b_n)_2$ are two initial rows of configurations of CA rule-j with L-T-R neighborhoods, then the corresponding IVT is $IVT_j^{2,3}(b', a, b'')$ where $b' = (b_nb_0b_1\ldots b_{n-1})_2$ and $b'' = (b_1b_2\ldots b_nb_0)_2$. This logic can be applied repeatedly for successive evolution. So pattern generated by CA rule-j is same as pattern generated by $IVT_j^{2,3}$ [13] as shown in Fig. 4.

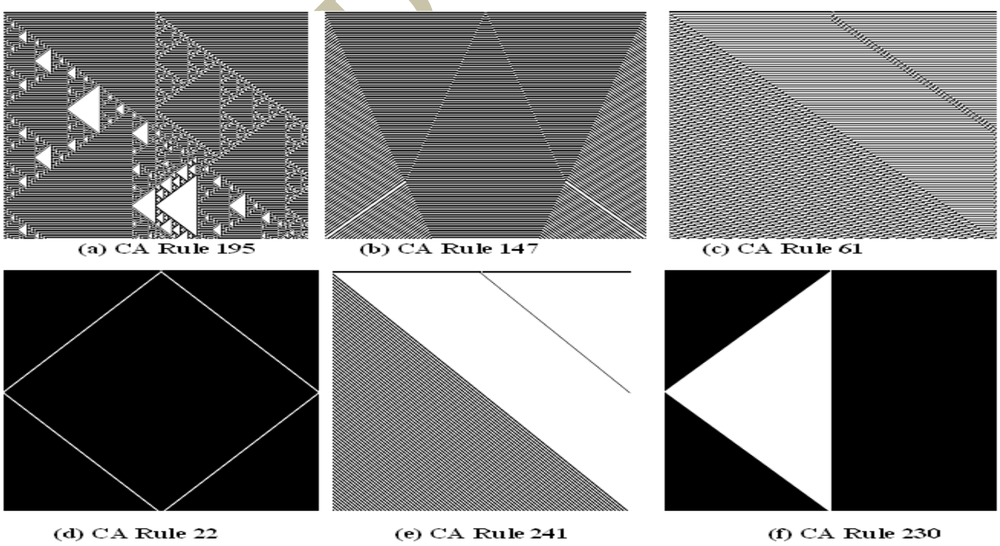


Fig. 4: (a) to (f) shows the space-time diagrams of CA Rules 195, Rule 147, Rule 61, Rule 22, Rule 241 and Rule 230 using L-T-R neighborhoods.

Similarly, for CA rule-j with T-RC-R neighborhoods, the corresponding IVT is $\mathrm{IVT}_j^{2,3}(a, a^/, b^/)$ where $a^/ = (a_0a_1\ldots a_{n-1}a_n)_2$ and $b^/ = (b_1b_2\ldots b_nb_0)_2$ and their patterns are shown in Fig. 5.

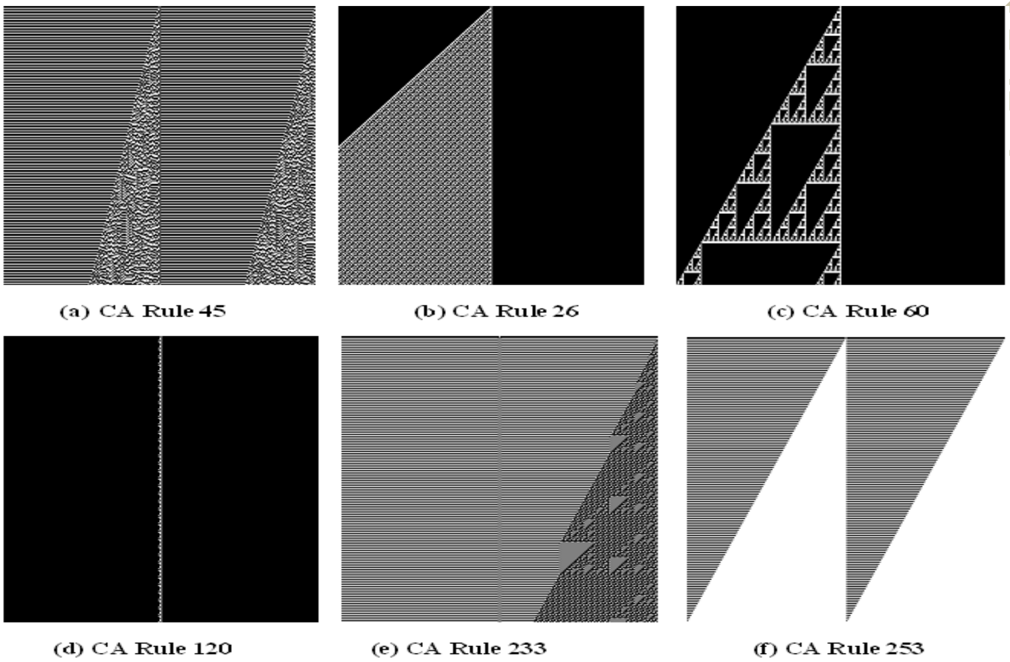


Fig. 5: (a) to (f) shows the space-time diagrams of CA Rules 45, Rule 26, Rule 60, Rule 120, Rule 233 and Rule 253 using T-RC-R neighborhoods.

As the equivalence of CA and IVTs are established based on their space-time diagrams so it will be interesting to explore number conserving patterns obtained by NCIVTs and NCCA rules.

7. Conclusion

The process of CA computation is similar to dynamic programming approach used in algorithms. As an example, for three neighborhood CA, the global problem is divided into overlapping local sub-problems of length 3. While computing locally, every cell value is used for three times to find the next configuration which requires to collect a total of 3n information although there are only n cells. On the contrary, the procedure of IVTs uses divide and conquer approach where the local sub-problems are independent of each other. So every cell value is used only once and hence a computation cost of n units is required while finding the next configuration.

In this paper, it has been shown that for every cellular automata (CA) rule, there exists one integral value transformation (IVT). It means there exists an one-to-one correspondence between NCCA and NCIVTs. Three types of NCIVTs, namely, uniform NCIVTs, non-uniform NCIVTs and non-uniform NCIVTs which may vary over time have been defined along with their cardinality in different dimensions. It has been observed that the cardinality of these NCIVTs increases exponentially as the length of binary string increases. Our future endeavor is to explore some important properties and applications of NCIVTs similar to NCCA rules in one and higher dimensions.

References

[1] Boccara, N. and Fukś, H. (1998), Cellular Automaton Rules Conserving the Number of Active Sites, Journal of Physics A: Mathematical and General, 31(28), 6007-6018.

[2] Boccara, N. and Fukś, H. (2002), Number Conserving Cellular Automaton Rules, Fundamenta Informaticae, 52(1-3), 1-13.

[3] Choudhury, P. P., Sahoo, S., Chakraborty, M., Bhandari, S. K. and Pal, A. (2009), Investigation of the global dynamics of cellular automata using Boolean derivatives, Computers & Mathematics with Applications, 57 (8), 1337-1351.

[4] Choudhury, P. P., Sahoo, S. and Chakraborty, M. (2008), Implementation of Basic Arithmetic Operations Using Cellular Automaton, ICIT-08, Proceedings of 11th International Conference on Information Technology, Bhubaneswar, India, 79-80.

[5] Choudhury, P. P., Sahoo, S., Nayak, B. K. and Hassan, Sk. S. (2011), Act of CVT and EVT In The Formation of Number-Theoretic Fractals, International Journal of Computational Cognition, 9(1), 1-8.

[6] Choudhury, P. P., Sahoo, S., Nayak, B. K. and Hassan, Sk. S. (2010), Theory of Carry Value Transformation and its Application in Fractal formation, Global Journal of Computer Science and Technology, 10(14), 98-107.

[7] Edwards, R. and Maignan, A. (2019), A class of discrete dynamical systems with properties of both cellular automata and l-systems, Natural Computing, 1–33.

[8] Hassan, Sk. S. (2015), Discrete dynamics of one dimensional Collatz like Integral Value Transformations, Journal of Applied Mathematics and Computing,49(1-2), 91-105.

[9] Hassan, Sk. S., Choudhury, P. P., Das, S., Singh, R. and Nayak, B. K. (2010), Collatz function like Integral Value Transformations, Alexandria Journal of Mathematics, 1(2), 30-35.

[10] Hattori, T. and Takesue, S. (1991), Additive conserved quantities in discrete-time lattice dynamical systems, Physica D. Nonlinear Phenomena, 49(3), 295-322.

[11] Neumann, J. von, Burks, A. W., (1966), Theory of self-reproducing automata, IEEE Transactions on Neural Networks, 42(4), 521-523.

[12] Pal, S., Sahoo, S. and Nayak, B. K. (2012), Properties of Carry Value Transformation, Int. J. of Mathematics and Mathematical Sciences, vol. 2012, 10 pages, DOI:10.1155/2012/174372.

[13] Sahoo, S., Das, S. and Pal, S. (2020), Patterns using Variations of Cellular Automata and Integral Value Transformations, International Conference on Advances in Computing, Communication & Materials (ICACCM 2020), DOI:10.1109/ ICACCM50413. 2020. 9212925.

[14] Sahoo, S., Hassan, Sk. S., Das, J. K. and Choudhury, P. P. (2020), Relationship of Two Discrete Dynamical Models: One-dimensional Cellular Automata and Integral Value Transformations, arXiv: 2006.13741v1[nlin.CG].

[15] Sahoo, S., Mohanty, I., Chowdhary, G. and Panigrahi, A. (2010), 2-Variable Boolean Operation-Its Use in Pattern Formation, arXiv: 1008.2530v1, nlin.CD.

[16] Sahoo, S., Pal, S. and Rout R. K. (2020),Visual Effects of Second Order Recurrence Equations through Space-Time Diagrams, International Conference on Industrial and Manufacturing Systems (CIMS-2020), Dr. B. R. Ambedkar National Institute of Technology, Jalandhar, Punjab, India. (In press)

[17] Schiff, J. L. (2011), Cellular automata: a discrete view of the world, 45, John Wiley & Sons.

[18] Ulam, S. (1962), On some mathematical problems connected with patterns of growth of figures, Proceedings of Symposia in Applied Mathematics, 14, 215–224.

[19] Wolfam, S. (2002), A New Kind of Science, Wolfram Media, Inc, Champaign, IL.

[20] Xu, X., Song, Y. and Banks, S. P. (2009), On the dynamical behavior of cellular automata, Int. J. of Bifurcation and Chaos, 19 (04), 1147–1156.