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A STUDY OF WEAK SOLUTION ON A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In the present paper, we focus on the existence theorem for weak solution for a class of Caputo fractional differential equation together with initial condition in continuous function space using Mönch's fixed-point theorem associated with the technique of measure of weak non-compactness. Further, the existence result of weak solution is extended in L_2 *space using Arzelà-Ascoli theorem.*

Keywords: Fractional derivatives and integrals, Fractional differential equation, Caputo derivative, Weak solution

MSC(2020): 26A33, 34A08, 34K37, 35D30.

1. Introduction

The subject fractional calculus is as old as the calculus [18]. It studies and analysis the integration and derivative of a functions to non-integer order. Due to various applicability of this subjects, many researchers' showed their interests in solving problems that arises in the fields of science and engineering (for instance, see

[5, 16, 17, 19, 23, 26, 30, 31] and the references therein). Especially, the authors in [7, 25, 28] use fractional calculus to understand and predict the transmission dynamics of COVID-19.

In recent years, the theory on existence and uniqueness of solutions of fractional differential equations have been studied by many authors [3, 6, 9, 14, 27, 32, 33] and the references therein.

In [29] discussed the existence of weak solution in a reflexive Banach space by considering the abstract Cauchy problem.

$$y' = f(t, y), \text{on } [0, T]$$

$$y(0) = x_0 \in E.$$

where $f : [0, T] \times E \to E$ is weakly-weakly continuous and E is a reflexive Banach space.

Further, Cramer et al. [11] extended the result to arbitrary Banach spaces. In [22], the author instigated an existence result for differential equation in Banach spaces relative to weak topology. There are a few results devoted to weak solutions of nonlinear fractional differential equations. The authors [8] investigated the existence of weak solution for fractional differential equations with mixed boundary value problem. In [4] presents a general result for the existence of weak solutions to fractional differential equations in non-reflexive Banach spaces. Later Abbas et al. [2] presented some results concerning the existence of weak solutions for some functional implicit differential equations of Hadamard fractional derivative. Recently, the study of weak solution for fractional differential equation attracted considerable amount of attention of many authors (see [13, 24, 32]).

Motivated by the work [1, 2], in this paper, we study the existence of weak solutions for a class of fractional differential equations with fractional weak derivative of the form

$$^{c}\mathcal{D}^{\alpha}u(t) = f\big(t, u(t),\ ^{c}\mathcal{D}^{\rho}u(t)\big);\ \ t \in I = [0, 1] \tag{1.1a}$$

with

$$u(0) = \phi. \tag{1.1b}$$

where $0 \leq \rho < \alpha \leq 1,\ T > 1, \phi \in E, f: I \times E \times E \to E$ is a given continuous function and E is a real(or complex) Banach space with Supremum norm.

2. Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the suprimum norm

$$\|v\|_\infty = \sup_{t\in I}\|v\|_E .$$

Also, let E^* be the topological dual of E. We denoting $AC(I)$ is the space of absolutely continuous functions from I into E.

Definition 1 (see [4]) A function $u(\cdot)$ is said to be weakly continuous at $t_0 \in I$ if for every $u^* \in E^*$, the scalar function $t \to \langle u^*, u(t)\rangle$ is continuous at t_0.

Definition 2 A function $h: E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).

Definition 3 (See [4]) A function $u(\cdot)$ is said to be weakly differentiable at $t_0 \in I$ if there exists an element $u'_w(t_0) \in E$ such that

$$\lim_{h\to 0}\langle u^*, \frac{u(t_0+h)-u(t_0)}{h}\rangle = \langle u^*, u'_w(t_0)\rangle$$

for every $u^* \in E$. The element $u'_w(t_0)$ will be also denoted by $\frac{d_w}{dt}u(t_0)$ and it is called the weak derivative of $u(\cdot)$ at $t_0 \in I$.

For more details we refer to [4, 21, 22] and reference therein.

A function $u(\cdot): I \to E$ is said Riemann-Pettis integrable (or RP-integrable) on I if $u(\cdot)$ is scalarly Riemann integrable and, for each interval $\subset I$, there exists an element $p \in E$ such that $\langle u^*, p\rangle = \int_J \langle u^*, p(\tau)\rangle\, d\tau$ for every $u^* \in E^*$.

It is well known that every R-integrable function is RP-integrable, and every RP-integrable function is Pettis integrable.

Definition 4 ([4]) Let $u(\cdot): I \to E$ be a RP-integrable on function define on I and $\alpha > 0$. Then the weak fractional Riemann-Liouville integral of order $\alpha > 0$ exists on I and is defined by

$$I^{\alpha}\, u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \qquad t \in I. \tag{2.1}$$

The fractional Caputo derivative of order $\alpha \in (0,1)$ is defined as

Definition 5 ([4]). Let $\alpha \in (0,1)$ and let $u(\cdot): I \to E$ be a weakly differentiable function on I If the derivative $u'(\cdot)$ of $u(\cdot)$ is RP-integrable on I, then

$${}^{c}\mathcal{D}^{\alpha} u(t) := I^{1-\alpha} u'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u'(s)\, ds, \qquad t \in I. \tag{2.2}$$

exists a.e. on I.

Lemma 1 ([4]). If $u(\cdot): I \to E$ is weakly differentiable a.e on I and $u'(\cdot)$ is RP-integrable on I and $\alpha \in (0,1)$, then

a) $I^{\alpha}\, {}^{c}\mathcal{D}^{\alpha} u(t) = u(t) - u(0)$ on I.
b) ${}^{c}\mathcal{D}^{\alpha} I^{\alpha}\, u\,(t) = u(t)$ on I.

Theorem 1 If $y(\cdot) \in RP(I, E)$, then the Abel integral equation

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds = y(t), \qquad t \in I. \tag{2.3}$$

has a solution in $u(\cdot) \in RP(I, E)$ if and only if the function $y_{1-\alpha}$ has the following properties:

a) $y_{1-\alpha}(\cdot)$ is wAC on I
b) $y_{1-\alpha}(\cdot)$ is weakly differentiable a.e on I and

$$u(t) = (y_{1-\alpha})'(t), \qquad \text{for a.e. } t \in I$$

c) $y_{1-\alpha}(0) = 0$,

where $(u_{1-\alpha})'(t)$ will be denoted by ${}^{RL}D^{\alpha} u(t)$ and is called the weak Riemann-Liouville derivative of $u(\cdot)$.

Lemma 2 Let $g(\cdot): I \to E$ be a weakly continuous functions. Then a continuous function $u(\cdot): I \to E$ is a weakly solution of

$${}^{c}\mathcal{D}^{\alpha} u\,(t) = g(t); \quad t \in I \tag{2.4a}$$

with

$$u(0) = \phi, \tag{2.4b}$$

is equivalent to the problem of obtaining the solution of the equation

$$u(t) = \phi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds. \tag{2.5}$$

Proof. First we consider $u(\cdot) : I \to E$ is a weak solution of (2.4a). Then by using lemma 1, we can easily derive (2.6).

Conversely, suppose that the continuous function $u(\cdot): I \to E$ satisfies the integral equation (2.6). Then (2.6) can be written as

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds = u(t) - \phi = v(t)$$

where $v(t) := u(t) - \phi$. By using Theorem 1 and [4, Remark 2.13], we get

$$g(t) = (v_{1-\alpha})'(t) = (u_{1-\alpha})'(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(0) \ for\ a.e\ t \in I$$

$$= {}^{c}\mathcal{D}^{\alpha} u\ (t).$$

where $v_{1-\alpha}(\cdot)$ is weakly differentiable a.e on I.

Theorem 2 (Lebesgue Dominance Theorem). Let (f_n) be a sequence of integrable functions which converges almost everywhere to a real valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and

$$\int f d\mu = \lim \int f d\mu.$$

Definition 6 (See [12]) Let E be a Banach space, Ω_E be the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak non-compactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$\beta(X) = \inf c > 0$: there exists a weakly compact $\Omega \subset E$ such that $X \subset cB_1 + \Omega$.

The De Blasi measure of weak non compactness satisfies the following properties [12]

1. $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
2. $\beta(A) = 0 \leftrightarrow$ A is weakly relatively compact,

3. $\beta(A \cup B) = \max\{(\beta(A), \beta(b)\}$,
4. $\beta(\bar{A}^{\omega}) = \beta(A)$, ($\bar{A}^{\omega}$ denotes the weak closure of A),
5. $\beta(A + B) \leq \beta(A) + \beta(B)$,
6. $\beta(\lambda A) = |\lambda|\beta(A)$,
7. $\beta(conv(A)) = \beta(A)$,
8. $\beta\left(\cup_{(|\lambda| \leq h)} \lambda A\right) = h\beta(A)$.

Preposition 1 (See [2]). Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $||\varphi|| = 1$ and $\varphi(x_0) = ||x_0||$.

For a given set V of functions $v: I \to$ E let us denote by

$$V(t) = \{v(t): v \in V\};\ t \in I$$

and

$$V(I) = \{v(t): v \in V, t \in I\}.$$

Lemma 3 (See [15]). Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and

$$\beta_C(H) = \max_{t \in E} \beta(H(t))$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds\,,$$

where $H(s) = \{u(s): u \in H, s \in I\}$, and β_C is the De-Blasi measure of weak non compactness defined on the bounded sets of C.

For our purpose, we need following the fixed-point theorem [21]:

Theorem 3 ([21]). Let Q be a non empty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose $T: Q \to Q$ is weakly sequentially continuous. If the implication

$$\overline{V} = \overline{conv}\left(\{0\} \cup T(V)\right) \Rightarrow V, \text{ is relatively compact,} \tag{2.6}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Lemma 4. If V is a strongly equicontinuous and uniformly bounded subset of $C_w(K,E)$, the space of weakly continuous functions $k \to E$ endowed with the topology of weak uniform convergence, then

(i) the function $t \to \beta(V(t))$ is continuous on K;

(ii) for each compact subset T of K

$$\beta\big(V(t)\big) = sup\{\,\beta\big(V(t)\big): t \in \ T\,\}.$$

3. Existence of weak solutions

Let us start by defining what we mean by a weak solution of the problem (1.1a)-(1.1b).

Definition 7. A measurable function $u \in \ C$ is said to be a weak solution of the problem (1.1a) if u satisfies the condition (1.1b) and the equation (1.1a) on I.

We establish our existence of weak solution two functional space. Which are discussed in the following subsections.

3.1 Weak Solutions in the Space of Continuous Functions

To establish our result concerning the existence of weak solutions (1.1a)-(1.1b), we list the following hypotheses:

$(H1)$ For a.e $t \in \ I$, the functions $v \to \ f(t,v,.)\ and\ w \to \ f(t,.,w)$ are weakly sequentially continuous.

$(H2)$ For each $v,w \in \ E$, the function $t \to \ f(t,v,w)$ is RP-integrable a.e on I.

$(H3)$ There exists $p \in \ C(I,[0,\infty))$ such that for all $\varphi \in \ E^\wedge *$, we have

$$\big|\varphi\big(f(t,v,w)\big)\big| \le \frac{p(t)\|\varphi\|}{1+\|\varphi\|+\ \|v\|_E\ +\|w\|_E}$$

for a.e $t \in \ I$ and each $v,w \in \ E$,

$(H4)$ For each bounded and measurable set $B \subset \ E$ and for each $t \in \ I$, we have

$$\beta(f(t,B,(\,{}^c\mathcal{D}^\rho\ B\,))\ \le\ p(t)\beta(B).$$

For any $p \in C(I, [0, \infty))$, we define

$$p^* = \sup_{t \in I} p(t) . \tag{2.7}$$

Theorem 4. Suppose that $f(\cdot,\cdot,\cdot): I \times E \times E \to E$ satisfies the assumption $(H_1) - (H_4)$ and if

$$L := \frac{p^*}{\Gamma(1+\alpha)} < 1. \tag{2.8}$$

Then the problem (1.1a) has atleast one solution defined on I.

Proof. First we convert the problem (1.1a)-(1.1b) into a fixed point problem by using Lemma 2 and consider the operator $T: C \to C$ defined by

$$(Tu)(t) = \phi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds , \tag{2.9}$$

where $g(\cdot) \in C$ with $g(t) = f(t, u(t), {}^c\mathcal{D}^\rho u(t))$.

From the assumption (H_2) it will directly imply that $t \mapsto (t-s)^{\alpha-1} g(s) ds$ for a.e. $t \in I$, is RP-integrable, and for each $u \backslash in\ C$, the function

$$t \mapsto f(t, u(t), {}^c\mathcal{D}^\rho u(t)),$$

is RP-integrable over I. Thus, the operator T is well defined. Let $R > 0$ be such that

$$R > \frac{p^*}{\Gamma(1+\alpha)}$$

And consider the set

$$Q = \left\{ u \in C: \|u\|_c \leq R \text{ and } \|(Tu)(t_2) - (Tu)(t_1)\|_E \right.$$
$$\leq \frac{p^*}{\Gamma(1+\alpha)} (t_2 - t_1)^\alpha$$
$$\left. + \frac{p^*}{\Gamma(\alpha)} \int_1^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \, ds \right\}.$$

Clearly, the subset Q is closed, convex and equicontinuous. We shall show that the operator T satisfies all the assumptions of Theorem 3. The proof will be given in steps.

Step 1. *T maps Q into itself.*

Let $u \in Q, t \in I$ and assume that $(Tu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|(Tu)(t)\|_E = \varphi((Tu)(t))$. Thus

$$\left\|(Tu)(t)\right\|_E = \varphi\left(\phi + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} g(s)\, ds\right)$$

where $g(\cdot) \in C$.

Since $\varphi \in E^*$, using the properties on dual space and (H_3) we get

$$\|(Tu)(t)\|_E \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} |\varphi(g(s))|\, ds = \frac{p^*}{\Gamma(1+\alpha)} t^\alpha \leq R.$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $u \in Q$ with

$$(Tu)(t_2) - (Tu)(t_1) \neq 0.$$

Then by Proposition 1 there exists $\varphi \in E^*$ such that

$$\|(Tu)(t_2) - (Tu)(t_1)\|_E = \varphi\big((Tu)(t_2) - (Tu)(t_1)\big)$$

and $||\varphi|| = 1$.

Hence

$$\begin{aligned}
\|(Tu)(t_2) - (Tu)(t_1)\|_E &= \varphi\left((Tu)(t_2) - (Tu)(t_1)\right) \\
&\leq \varphi\left(\left(\phi + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1}g(s)ds\right)\right. \\
&\quad \left. - \left(\phi + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2 - s)^{\alpha-1}g(s)ds\right)\right) \\
&\leq \int_{t_1}^{t_2}|t_2 - s|^{\alpha-1}\frac{\left|\varphi\left(g(s)\right)\right|}{\Gamma(\alpha)}ds \\
&\quad + \frac{p^*}{\Gamma(\alpha)}\int_0^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|(Tu)(t_2) - (Tu)(t_1)\|_E \\
&\leq \frac{p^*}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}\,ds + \int_0^{t_1}|(t_2 - s)^{\alpha-1} \\
&- (t_1 - s)^{\alpha-1}|\frac{p(s)}{\Gamma(\alpha)}ds \\
&= \frac{p^*}{\Gamma(\alpha+1)}(t_2 - t_1)^{\alpha} + \frac{p^*}{\Gamma(\alpha)}\int_0^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|ds.
\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. *T is weakly-sequentially continuous.*

Let (u_n) be a sequence in Q and let $u_n(t) \to u(t)$ in (E, w) for each $t \in I$. Since$f$ satisfies the assumption (H_1), we have $f(t, u_n(t), {}^cD_\rho^n\ (t)$ converges weakly uniformly to $f(t, u(t), {}^cD_\rho^n\ (t))$.

Then there exist $\varphi \in E^*$ such that

$$(Tu)(t) = \phi + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds$$

where $g(\cdot) \in C$ with $g(t) = f(t, u(t), {}^cD_\rho^n \ (t)$.

Then

$$\left|\left|(Tu)(t)\right|\right|_E \leq \varphi\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds\right)$$

$$= \frac{1}{\Gamma(\alpha)} h(s),$$

where $h(s) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\varphi\big(g(s)\big)\right| ds$.

Clearly, the function $h(s)$ is continuous as well as bounded as we have $\varphi \in E^*$ and $t \in (0,1)$. Therefore the function $h(s)$ is Lebesgue integrable. Hence by Lebesgue dominated convergence theorem Theorem 2 for RP-integrable implies $T(u_n) \to T(u)$. Hence $T\colon Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (2.6) holds.*

Let V be a subset of Q such that $\overline{U} = \overline{conv}(T(U) \cup \{0\})$. Obviously

$$U(t) \subset \overline{conv}\Big(\big((NU)(t)\big) \cup \{0\}\Big), t \in I.$$

Further, as V is bounded and equicontinuous. by [10] and Lemma 2 the function $t \to u(t) = \beta(U(t))$ is continuous on I. From (H_3), (H_4), Lemma 1 and the properties of the measure β, for any $t \in I$, we have

$$u(t) \leq \ \& \beta\big((TU)(t) \cup \{0\}\big) \leq \beta((TU)(t))$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} p(s)\beta(U(s))\, ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t p(s)u(s)(t-s)^{(\alpha-1)} ds$$

$$\leq \frac{p^*}{\Gamma(\alpha+1)}\|u\|_C.$$

Thus

$$\|u\|_C \leq L\|u\|_C .$$

From (2.7), we get $\|u\|_C = 0$, that is $u(t) = \beta(U(t)) = 0$, for each $t \in I$. and then by [20, Theorem 2], U is weakly relatively compact in C Applying now Theorem 3, we conclude that T has a fixed point which is a solution of the problem (1.1a)-(1.1b).■

3.2 Weak solution in L_2-space

In this subsection we will established our results in L_2-space for the fractional differential equations (1.1a)-(1.1b). To establish our results we use Arzelà-Ascoli theorem.

Lemma 5 (Arzelà-Ascoli). Let $D \subset R^n$ be compact and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on D If $F = \{f_n : n \in N\}$ is uniformly bounded and equicontinuous on D then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges uniformly to a function $f \in C(D, R)$.

Equation (1.1a)-(1.1b) reduced to the integral form as

$$u(t) = \phi + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(t, u(t), ({}^{c}\mathcal{D}^{\rho} u)(t)) ds, \tag{3.1}$$

where $\alpha \in (0,1)$ and $t \geq t_0$. Next, the local existence theorem of the IVP (1.1a)-(1.1b) is given below.

$$J = [t_0 - a, t_0 + a], B = \{u \in \mathbb{R}^d | \ \|u - \phi\| \leq b\},$$

$$E = \{(t, u) \in \mathbb{R} \times \mathbb{R}^d | \ t \in J, u \in B\}.$$

Theorem 5. Assume that the function $f: E \to R^d$ satisfies the following conditions:

1. $f(t, u, {}^{c}\mathcal{D}^{\rho} u)$ is Lebesgue measurable with respect to t on J;
2. $f(t, u, {}^{c}\mathcal{D}^{\rho} u)$ is continuous with respect to u on B;
3. There exists a real valued function $m(t) \in L^2(J)$ such that
$$\|f(t, u, {}^{c}\mathcal{D}^{\rho} u)\| \leq m(t)$$
for almost every $t \in J$and all $u \in B$.

Then, for $\alpha > \frac{1}{2}$ there at least exists a solution of the IVP (1.1a)-(1.1b) on the interval $[t_0 - h, t_0 + h]$ for some positive number h.

Proof. Here the initial value problem (1.1a)-(1.1b) for the case where $t \in [t_0, t_0 + h]$ are only discussed. The proof is divided into three steps.

Step 1. In the first step it is to prove that $(t-s)^{\alpha-1} f(s, u(s), {}^c\mathcal{D}^\rho u(s))$ is Lebesgue integrable with respect to $s \in [t\_0, t]$ $(t \le t_0 + h \le a)$ for all t in between J provided that $u(s)$ is Lebesgue measurable on the interval $[t_0, t_0 + h]$. When $u(t)$ is chosen to be a constant vector, that is $u(t) = c(t_0 \le t \le t_0 + h)$, $f(t, c)$ is Lebesgue measurable due to condition (1)

Generally, for any Lebesgue measurable $u(t)$ on $[t_0, t_0 + h]$, there exists a sequence of step functions, denoted by $\{u_k(t)\}(k = 1, 2, \cdots)$, such that $u_k(t)$ is convergent to $u(t)$ almost everywhere as $k \to \infty$. Consequently, the limit function $f(t, u(t), {}^c\mathcal{D}^\rho u(t))$ is Lebesgue measurable on $[t_0, t_0 + h]$,

Moreover, it follows from condition (3) that

$$\|(t-s)^{\alpha-1} f(t, u(t), {}^c\mathcal{D}^\rho u(t))\| \le (t-s)^{\alpha-1} m(s) \tag{3.2}$$

for almost every $s \le t$ with $s, t \in J$. It is observed that $(t-s)^{\alpha-1} \in L^2[t_0, t]$ if $\alpha > \frac{1}{2}$. In the light of Hölder inequality, we obtain that $(t-s)^{\alpha-1} f(s, u(s), {}^c\mathcal{D}^\rho u(s))$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all t, and

$$\int_{t_0}^{t} \|(t-s)^{\alpha-1}(t, u(t), {}^c\mathcal{D}^\rho u(t))\| ds \le \|(t-s)^{\alpha-1}\|_{L^2[t_0,t]}, \tag{3.3}$$

where $\|H(t)\|_{L^P[I]} = \left(\int_I |H(s)|\right)^{\frac{1}{P}}$ for any L^P -integrable function $H: I \to \mathbb{R}$. This completes Step 1.

Step 2. Here, a sequence of vector-valued functions are constructed and their uniform boundedness and equicontinuity are verified, respectively.

Note the completely continuity of the function $m^2(t)$ implies that for a given positive number M, there must exist a number $h' > 0$ satisfying

$$\int_{t_0}^{t_0+h} m^2(s)ds \leq M \tag{3.4}$$

whenever $h < \min\left\{h', a, \left[\frac{b^2\Gamma^2(\alpha)(2\alpha-1)}{M}\right]^{\frac{1}{2\alpha-1}}\right\}$. Once we select a proper h , a sequence of vector-valued functions, denoted by $\{u_n(t)\}_{n=1}^{\infty}$, can be defined as

$$u_n(t) = \begin{cases} \phi, & t_0 \leq t \leq t_0 + \frac{h}{n} \\ \phi + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t-\frac{h}{n}} (t-s)^{\alpha-1} f(s, u_n(s), D^\rho u_n(s))ds, & t_0 + \frac{h}{n} \leq t \leq t_0 + h \end{cases}.$$

From the above definition, when$t_0 + \frac{h}{n} \leq t \leq t_0 + \frac{2h}{n}$, one has $t_0 \leq t - \frac{h}{n} \leq t_0 + \frac{h}{n}$, which implies that $f(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s)) \equiv f(s, \phi)$ as $t_0 \leq s \leq t - \frac{h}{n}$. Therefore, $f(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s))$ is Lebesgue measurable and $(t - s)^{\alpha-1} f(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s))ds$ is Lebesgue integrable on $\left[t_0, t - \frac{h}{n}\right]$.

Next, we shall prove that $u_n(t)$ is continuous on $\left[t_0, t_0 + \frac{2h}{n}\right]$ for all n. It is obvious that $u_n(t)$ is continuous on $\left[t_0, t_0 + \frac{h}{n}\right]$ for all n. If additionally the interval $\left[t_0 + \frac{h}{n}, t_0 + \frac{2h}{n}\right]$ is taken into account, two cases are discussed respectively.

***Case* A**. When $t_0 \leq t_1 \leq t_0 + \frac{h}{n} < t_2 \leq t_0 + \frac{2h}{n}$, it follows from (3.2) and (3.3) that

$$\begin{aligned} \|u_n(t_2) - u_n(t_1)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2-\frac{h}{n}} (t_2 - s)^{\alpha-1} \left\|f\left(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s)\right)\right\| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2-\frac{h}{n}} (t_2 - s)^{\alpha-1} m(s)ds \\ &\leq \frac{1}{\Gamma(\alpha)} \sqrt{\frac{M}{2\alpha-1}} \left[t_0 - \left(t_0 + \frac{h}{n}\right)\right]^{\alpha-\frac{1}{2}}. \end{aligned}$$

Hence for any positive number $\delta < \left[\frac{\epsilon^2\Gamma^2(\alpha)(2\alpha-1)}{M}\right]^{\frac{1}{2\alpha-1}}$ such that for all $t_2 - \left(t_0 + \frac{h}{n}\right) \leq t_2 - t_1 \leq \delta$ and for all n.

***Case* B**. When $t_0 + \frac{h}{n} \leq t_1 \leq t_2 \leq t_0 + \frac{2h}{n}$, one has

$$I_1 = \int_{t_0}^{t_1 - \frac{h}{n}} [\,(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\,]^2\, ds = \frac{1}{2\alpha - 1}(S_1 + S_2)$$

where

$$S_1 = |(t_1 - t_0)^{2\alpha-1} - (t_2 - t_0)^{2\alpha-1}|,$$

$$S_2 = \left(t_2 - t_1 + \frac{h}{n}\right)^{2\alpha-1} - \left(\frac{h}{n}\right)^{2\alpha-1}.$$

Thereby, substitution of these inequalities into the following estimation gives that for all $t_2 - t_1 \leq \overline{\delta} = min\,\{\delta_1, \delta_2, \delta_3\}$,

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1 - \frac{h}{n}} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \left\| f\left(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s)\right) \right\| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_1 - \frac{h}{n}} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] m(s) ds < \frac{\epsilon}{2}.$$

Meanwhile, for all $t_2 - t_1 \leq \overline{\delta}$, it yields

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2 - \frac{h}{n}} (t_2 - s)^{\alpha-1} \left\| f\left(s, u_n(s), {}^c\mathcal{D}^\rho u_n(s)\right) \right\| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2 - \frac{h}{n}} (t_2 - s)^{\alpha-1} m(s) ds < \frac{\epsilon}{2\sqrt{2}}.$$

This consequently implies that whenever $t_2 - t_1 \leq \overline{\delta}$,

$$\|u_n(t_2) - u_n(t_1)\| \leq I_2 + I_3 < \epsilon.$$

Therefore, the above performed two cases leads to a conclusion that $u_n(t)$ is continuous with respect to t on $\left[t_0, t_0 + \frac{2h}{n}\right]$ for all positive integers n.

On the other hand, one has that for all $t \in \left[t_0, t_0 + \frac{h}{n}\right]$,

$$\|u_n(t) - \phi\| = 0,$$

and for all $t \in \left[t_0 + \frac{h}{n}, t_0 + \frac{2h}{n}\right]$,

$$\|u_n(t) - \phi\| = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t-\frac{h}{n}} (t-s)^{\alpha-1} m(s) ds$$

$$\leq \sqrt{\frac{M}{2\alpha - 1}} h^{\alpha - \frac{1}{2}} \leq \ b,$$

which implies that $f\left(t, u_n(t), {}^c\mathcal{D}^\rho u_n(t)\right) \in E$ for all n.

By induction, we can contend that the function $\{u_n(t)\}$ is continuous with respect to t on $[t_0, t_0 + h]$, satisfying $f\left(t, u_n(t), {}^c\mathcal{D}^\rho u_n(t)\right) \in E$ for all n In fact, it can be assumed that for a given integer k and all $0 \leq \ i < k < n, u_n(t)$ is continuous on $\left[t_0 + \frac{ih}{n}, t_0 + \frac{(i+1)h}{n}\right]$ and $\|u_n(t) - \phi\| \leq b$ for all n. Note that if $t \in \left[t_0 + \frac{kh}{n}, t_0 + \frac{(k+1)h}{n}\right]$. Then, by using similar argument performed above, it can be concluded that $u_n(t)$ is continuous on $\left[t_0 + \frac{kh}{n}, t_0 + \frac{(k+1)h}{n}\right]$ and $\|u_n(t) - \phi\| \leq b$ for all n.

Step 3. Using The Arzelà-Ascoli lemma and the conclusion derived in step 2, there must exist a subsequence $\{u_{n_k}(t)_{k=1}^{\infty}\} \triangleq \{u_k(t)_{k=1}^{\infty}\}$ contained in $\{u_n(t)_{n=1}^{\infty}\}$ such that $\{u_k(t)_{k=1}^{\infty}\}$ is uniformly convergent to $u(t)$ which is continuous with respect to t on $[t_0, t_0 + h]$. So, in what follows it is to prove that this limit function $u(t)$ is a solution of equation (1.1a).

It follows from condition (2) that for any positive ζ there exist $N_1 \in N$, such that for all $k > N_1$,

$$\left\|f\left(t,u_k(t)\right)-f\ (t,u(t))\right\| < \frac{\Gamma(\alpha\ +1)\zeta}{2h^\alpha}$$

due to condition (2). Now, we prove that $u(t)$ satisfies (1.1a). We have that

$$I_4=\left\|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t-\frac{h}{k}}(t-s)^{\alpha-1}f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))ds\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))ds\right\|$$

$$=\frac{1}{\Gamma(\alpha)}\left\|\int_{t_0}^{t}(t-s)^{\alpha-1}\left[f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))-f\left(s,u(s),\ {}^cD^\rho u(s)\right)\right]ds\right.$$

$$\left.-\int_{t-\frac{h}{k}}^{t}(t-s)^{\alpha-1}f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))ds\right\|$$

$$\leq\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left\|f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))-f\left(s,u(s),\ {}^c\mathcal{D}^\rho u(s)\right)\right\|ds$$

$$+\int_{t-\frac{h}{k}}^{t}(t-s)^{\alpha-1}\|f(s,u_k(s),\ {}^c\mathcal{D}^\rho u_k(s))\|ds$$

$$\triangleq I_5+I_6.$$

Using (1.1a), one obtains

$$I_5=\frac{\Gamma(\alpha+1)\zeta}{2h^\alpha}\frac{h^\alpha}{\Gamma(\alpha+1)} < \frac{\zeta}{2}.$$

Also there exists a natural number $N_2=\left[\left(\frac{2b}{\zeta}\right)^{\frac{2}{2\alpha-1}}\right]$ such that for all $k > N_2$,

$$I_6 \leq \frac{1}{\Gamma(\alpha)} \int_{t-\frac{h}{k}}^{t} (t-s)^{\alpha-1} \, m(s) ds \; \leq \frac{1}{\Gamma(\alpha)} \sqrt{\frac{M}{2\alpha - 1}} \left(\frac{h}{k}\right)^{\alpha-1} \; < \; \frac{\zeta}{2},$$

Hence, setting $N = \max\{N_1, N_2\}$, one arrives at $I_4 < \zeta$ for all $k > N$.

Consequently, $u(t)$ satisfies

$$u(t) = \phi + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, u_k(s), ({}^{c}\mathcal{D}^{\rho} u_k)(s)) ds.$$

This implies that there at least exists a solution of (3.1) on $[t_0, t_0 + h]$. Hence, the initial value problem (1.1a)-(1.1b) has at least exists a solution on the interval $[t_0, t_0 + h]$. Moreover, the similar arguments could be applied to obtain a solution of the initial value problem 1.1a)-(1.1b) on $[t_0 - h, t_0]$. This finally completes the proof.■

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