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AXIOMATIZATION OF THE INTERVAL VALUED SHAPLEY FUNCTION ON A CLASS OF COOPERATIVE INTERVAL GAMES WITH FUZZY COALITIONS

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Abstract

In this paper we focus on the fuzzy Cooperative games with interval uncertainty, interval games with fuzzy Coalitions. A set of axioms to characterize the interval valued Shapley function for an interval game with fuzzy Coalitions is proposed. We formulated a specific expression of the interval valued Shapley function in the class of interval fuzzy games namely the fuzzy cooperative interval game in Choquet integral form. Finally an example is given.

Keywords: Cooperative interval game, fuzzy coalition, fuzzy cooperative interval game, Shapley function, Choquet integral.

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1. Introduction

A cooperative interval game is a pair (N, v), where $N = \{1,2, \ldots, n\}$is a finite set of players and v, a characteristic function defined on 2^N that assigns every crisp subset (coalition) a compact interval in $\mathbb{R}$ called its worth interval (or worth set) giving $[0,0]$ worth to the empty coalition. The model of cooperative interval games with interval uncertainty of coalition values is an extension of the model of classical cooperative game with transferable utilities. We recall that a cooperative game with transferable utilities or simply a TU game $(N, w)$is defined by $w : 2^N \to \mathbb{R}$ with $w(\emptyset) = 0$.The cardinality of any coalition by $S \in 2^N$by $|S|$. In standard articulations, the worth or power $v(S)$is the amount of money or utility that the coalition S generates by means of cooperation. Solutions for interval cooperative games are interval payoff vectors, i.e., vectors whose components belong to $I(\mathbb{R})$ the set of all nonempty compact interval in $\mathbb{R}$. Interval solutions are effective to solve reward/cost sharing problems with interval uncertainty data, economic situations with interval uncertainty data, real life situation where businesses face interval uncertainty in decision making regarding cooperation, sociology, computer science, etc. Interval Shapley function is a significant interval solution for cooperative interval games introduced by Alparslan Gök et al. (2010).

Cooperative games with fuzzy coalitions or simply fuzzy games are generalization of crisp TU games in the sense that participation of the players is considered here partial that ranges between 0 and 1, see Aubin (1988). A fuzzy coalition is a fuzzy subset of the player set N, i.e., a function from N to [0,1]. The class of all fuzzy subsets (or fuzzy coalitions) is denoted by $L(N)$.

Besides the definition and axiomatization of cooperative interval games under fuzzy environment, in this paper, we introduce interval valued solution concepts for cooperative interval games with fuzzy coalition namely the interval valued Shapley function. We formulate the interval valued Shapley function in Choquet integral form.

The rest of the paper proceeds as follows. Section 2 compiles the related definitions and results pertaining to the development of the paper. Section 3 discussed the notion of cooperative interval games with fuzzy coalition. Section 4 introduces the notion of interval valued Shapley function for fuzzy cooperative interval games with Choquet integral form. In Section 4 we also obtain the interval solution vector of an example of interval fuzzy game.

2. Preliminaries

In this section we recall basic concepts definitions and results from Alparslan(2010), Alparslan (2015), Tsurumi(2001) and Biswakarma(2018) relevant to the development of the paper for ready reference.

2.1Cooperative interval games and interval valued Shapley function under crisp environment

Defination 1. Let $N = \{1,2,\dots,n\}$be the finite set n of players. Let 2^N denote the power set of N. A cooperative interval game is an ordered pair $(N, v)$where $v : 2^N \rightarrow I(\mathbb{R})$ is a characteristic function on N such tha $v(\emptyset) = [0,0]$, where $I(\mathbb{R})$is the set of all nonempty compact interval in $\mathbb{R}$.

Eachnon-empty coalition$S \in 2^N$ the value $v(S) = [\underline{v}(S), \overline{v}(S)]$ is called the worth interval of the coalition S in the interval game (N, v). The lower bound $\underline{v}(S)$is called the minimal worth which coalition S could receive on its own and the upper bound$\overline{v}(S)$is called the maximal worth which coalition S could get. The family of all interval games with player set N is denoted by $IG(N)$.

Result 1. If $v_1, v_2 \in IG(N)$, $S \in 2^N$ and $v_1(S) = [\underline{v_1}(S), \overline{v_1}(S)]$, $v_2(S) = [\underline{v_2}(S), \overline{v_2}(S)] \in I(\mathbb{R})$then $v_1(S) + v_2(S) = [\underline{v_1}(S), \overline{v_1}(S)] + [\underline{v_2}(S), \overline{v_2}(S)] = \left[\underline{v_1}(S) + \underline{v_2}(S), \overline{v_1}(S) + \overline{v_2}(S)\right] = \left[\left(\underline{v_2} + \underline{v_2}\right)(S), (\overline{v_1} + \overline{v_2})(S)\right] = (v_1 + v_2)(S) \in I(\mathbb{R}) \Rightarrow v_1 + v_2 \in IG(N)$.

Result 2. If $v_1 \in IG(N)$, $S \in 2^N$, $\lambda \in \mathbb{R}_+$ and $v_1(S) = [\underline{v_1}(S), \overline{v_1}(S)] \in I(\mathbb{R})$then$(\lambda v_1)(S) = \left[\left(\lambda \underline{v_1}\right)(S), (\lambda \overline{v_1})(S)\right] = \left[\lambda\left(\underline{v_1}\right)(S), \lambda(\overline{v_1})(S)\right] = \lambda\left[\underline{v_1}(S), \overline{v_1}(S)\right] = \lambda(v_1)(S) \in I(\mathbb{R}) \Rightarrow \lambda v_1 \in IG(N)$.

So from result 1. and result 2. we can conclude that $IG(N)$ has a cone structure with respect to addition and multiplication with non-negative scaler described above. The substraction of two interval game existi.e.,$v_1 + v_2 \in IG(N)$ if $|(v_1)(S)| \geq |(v_2)(S)|$for $v_1, v_2 \in IG(N)$, $S \in 2^N$.

Defination 2. An ordered pair $(N, |v|)$ is called a length game if $|v|(S) = \overline{v}(S) - \underline{v}(S)$for each $S \in 2^N$ and $v(S) = \left[\underline{v}(S), \overline{v}(S)\right] \in I(\mathbb{R})$.

Defination 3. A cooperative interval game (N, v) is called a size monotonic game if the length game $(N, |v|)$ is monotonic I.e., $|v|(S) \le |v|(T)$ for all $S, T \in 2^N$ with $S \subseteq T$.

For further use$SMIG(N)$ denotes the set of all size monotonic cooperative interval games.

Defination 4. The unanimity game with respect to $S \in 2^N, S \neq \emptyset$ denoted by u_S, is defined by

$$u_S(T) = \begin{cases} 1, & if\ T \supseteq S \\ 0, & otherwise \end{cases}$$

For a given number of n players, the set of all n-person TU games is denoted by $G(N)$. The set $G(N)$ is a linear space under the addition and scalar multiplications of functions given by

$(v_1 + v_2)(K) = v_1(K) + v_2(K)$ and $(\alpha v_1)(K) = \alpha. v_1(K) \alpha \in (\mathbb{R}), K \in 2^N$

The family of unanimity games is a basis for the linear space $G(N)$.

Defination 5. Let $S \in 2^N \setminus \{\emptyset\}$, $I \in I(\mathbb{R})$ and let u_S be the unanimity game based on S. The cooperative interval game (N, Iu_S) is defined by $(Iu_S)(T) = u_S(T)I = I$ if $T \supseteq S$ and $(Iu_S)(T) = u_S(T)I = [0,0]$ otherwise.

In the sequel of such interval games will play a central role. We denote by $\Gamma IG(N) \subset SMIG(N)$ the additive cone generated by the set $\Gamma = \{Iu_S : S \in 2^N \setminus \{\emptyset\}, I_S \in I(\mathbb{R})\}$. So each element of the cone is the finite sum of element of Γ i.e., for every$v \in \Gamma IG(N), v = \sum_{\emptyset \neq T \in 2^N} I_S u_S$.

Defination 6. Let $v \in IG(N)$ and $K \in 2^N$, $S \in 2^K$ is called a carrier in a coalition $K$for an interval game v if $v(S \cap T) = v(T), \forall\, T \in 2^K$.

Defination 7. An interval payoff vector for the player set $N = \{1,2, \dots, n\}$ is $(I_1, I_2, \dots, I_n)$, where $I_i \in I(\mathbb{R})$, $i \in N$.The set of such interval payoff vectors is denoted by $I(\mathbb{R})^N$.

Defination 8. If $\Pi(N) = \{\pi : \pi \text{ } is \text{ } a \text{ } permutation \text{ } of \text{ } N\}$ and the set of predecessors of i in π is $P_\pi(i) = \{j : \pi^{-1}(j) < \pi^{-1}(i)\}$. Then the interval marginal vector $m^\pi(v)$ of $v \in SMIG(N)$ with respect to π is

$$m^\pi(v) = v(P_\pi(i) \cup \{i\}) - v(P_\pi(i)) \quad for \text{ } each \text{ } i \in N$$

Defination 9. A function $\varphi: IG_0(N) \subseteq IG(N) \to I(\mathbb{R})^N$ is said to be a interval valuedShapley function on $IG_0(N)$ if it satisfies the following four axioms.

Axiom S$_1$. If $v \in IG_0(N)$ and $T \in 2^N$ then

$$\sum_{i \in W} \varphi_i(T, v) = v(T)$$

$$\varphi_i(T, v) = 0 \quad \forall \quad i \notin T,$$

Where $\varphi_i(T, v)$ is the ith interval of $\varphi_i(T, v) \in I(\mathbb{R})^N$.

Axiom S$_2$. If $v \in IG_0(N)$, $T \in 2^N$ and$S$ is a carrier in$T$then

$$\varphi_i(T, v) = \varphi_i(S, v) \quad \forall \text{ } i \in N,$$

Axiom S$_3$. If $v \in IG_0(N)$, $K \in 2^N$ and $i, j \in T$ are symmetric i.e., $v(K \cup i) = v(K \cup j)$ holds for any $S \in 2^{K \setminus \{i,j\}}$, then

$$\varphi_i(K, v) = \varphi_j(K, v)$$

Axiom S$_4$. For any $v_1, v_2 \in IG_0(N) \Rightarrow v_1 + v_2 \in IG_0(N)$ and satisfies$(v_1 + v_2)(T) = v_1(T) + v_2(T)$ for each $T \in 2^N$. If $v_1, v_2 \in IG_0(N)$ and $T \in 2^N$, then

$$\varphi_i(T, v_1 + v_2) = \varphi_i(T, v_1) + \varphi_i(T, v_2) \forall \text{ } i \in N$$

Theorem 3. *Define a function* $\varphi: SMIG(N) \subseteq IG(N) \to I(\mathbb{R})^N$ *by*

$$\varphi_i(K,v) = \begin{cases} \sum_{\{T\in 2^K | i\notin T\}} \frac{(|T|)!\,(|K|-|T|-1)!}{|K|!} \{v(T\cup\{i\}) - v(T)\}, & if\ i \in K \\ 0, & otherwise \end{cases}$$

Then the function φ is the interval valued Shapley function on $SMIG(N)$.

Proof. Let $v \in IG(N)$ and $K \in 2^N$. Then $\varphi_i(K,v)$ can be obtained in the same manner as the interval Shapley value, namely $\varphi_i(N,v)$. It was proved for $\varphi_i(N,v)$ in [1].

Theorem 3. *Let $\varphi: \Gamma IG(N) \subseteq SMIG(N) \to I(\mathbb{R})^N$ is the unique interval valued Shapley function satisfying the Axioms* **S_1-S_4** *on the class $\Gamma IG(N)$.*

Proof. From Theorem 4.1 of [1] it is not difficult to get the result.

2.2 Cooperative games with fuzzy coalition

In this section, we introduce some basic concepts and notions of a cooperative game with fuzzy coalitions or simply a fuzzy game. For a finite set of players $N = \{1,2,\ldots,n\}$ a fuzzy coalition variable is denoted by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in [0,1]$, and we will call $\boldsymbol{x}^f = \{\boldsymbol{y}: y_i = x_i\ or\ y_i = 0\ for\ each\ i \in N\}$ the set of all fuzzy coalition variable is created by $\boldsymbol{x}$. The n-vectors $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is called a fuzzy coalition whose components$s_i$is a constant which denotes the participation level of player i for the named fuzzy coalition. Similar to $\boldsymbol{x}^f$, we denotes $\boldsymbol{s}^f = \{\boldsymbol{t}: t_i = s_i\ or\ t_i = 0\ for\ each\ i \in N\}$ the set of all fuzzy coalition created by $\boldsymbol{s}$. For a fuzzy coalition $\boldsymbol{s}$the level set is denoted by $[\boldsymbol{s}]_h = \{i \in N: s_i \geq h\}$ for any $h \in [0,1]$, and the support is denoted by $Supp(\boldsymbol{s}) = \{i \in N: s_i > 0\}$. The class of all fuzzy subset of a fuzzy set $\boldsymbol{s} \subseteq N$ is denoted by $L(\boldsymbol{s})$.

In this paper, we use the notation $\boldsymbol{s} \subseteq \boldsymbol{t}$if and only if$s_i = t_i\ or\ s_i = 0$ for all $i \in N$ and $\boldsymbol{s} \leq \boldsymbol{t}$if and only if $s_i \leq t_i$ for all $i \in N$. For any $\boldsymbol{s}, \boldsymbol{t} \in L(N)$, union and intersection of two fuzzy coalition is denoted by $\boldsymbol{s}$ and $\boldsymbol{t}$are defined as usual i.e., $\boldsymbol{s} \vee \boldsymbol{t} = \boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ with $p_i = \max\{s_i, t_i\}, \forall i \in N$ and $\boldsymbol{s} \wedge \boldsymbol{t} = \boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ with $p_i = \min\{s_i, t_i\}, \forall i \in N$, respectively. Now we will introduce the following fuzzy set originally introduced by

Tsurumi et al. (2010). Let $\boldsymbol{s} \in L(N)$ and $i, j \in N$. For any $\boldsymbol{t} \in \boldsymbol{s}^f$, define $\boldsymbol{\Upsilon}^{ij}[\boldsymbol{t}] = (\Upsilon_1^{ij}(\boldsymbol{t}), \Upsilon_2^{ij}(\boldsymbol{t}), \dots, \Upsilon_n^{ij}(\boldsymbol{t})) \in L(\boldsymbol{s})$ by

$$\boldsymbol{\Upsilon}^{ij}[\boldsymbol{t}] = \begin{cases} t_j, & if\ k = i \\ t_i, & if\ k = j \\ t_k, otherwise \end{cases}$$

Defination 10. A cooperative game with fuzzy coalitions or simply a fuzzy cooperative game is a function $w : L(N) \to \mathbb{R}_+ \cup \{0\} = \{r \in \mathbb{R}: r \geq 0\}$ which assigns $w(\emptyset) = 0$. $FG(N)$ denote the class of all fuzzy games with player set N.

3. Cooperative interval games with fuzzy coalition

Defination 11. A cooperative interval games with fuzzy coalitions or simply a fuzzy cooperative interval game is a function $v : L(N) \to I(\mathbb{R}_+)$ is such that$v(\emptyset) = [0,0]$, where$\mathbb{R}_+ = \{r \in \mathbb{R}: r \geq 0\}$ and $I(\mathbb{R}_+)$ is the set of all compact interval in $\mathbb{R}_+$. $IFG(N)$ denote the set of all fuzzy cooperative interval games.

Defination 12. Let $v \in IFG(N)$ and $\boldsymbol{s} \in L(N)$, based on $\boldsymbol{s}$, a fuzzy coalition $\boldsymbol{s}' \in \boldsymbol{s}^f$ is called a carrier of $\boldsymbol{s}$in v if for any $\boldsymbol{t} \in \boldsymbol{s}^f, v(\boldsymbol{t}) = v(\boldsymbol{t} \wedge \boldsymbol{s})$.

All set of carriers of $\boldsymbol{s}$in v is denoted by $\boldsymbol{s}$in $S_C(v, \boldsymbol{s})$. Let $i \in N$, and $\boldsymbol{s} \in L(N)$. Define a fuzzy coalition $\boldsymbol{s}^i = (s_1^i, s_2^i, \dots, s_n^i) \in L(N)$ by

$$s_j^i = \begin{cases} s_i, & if\ i = j \\ 0, & otherwise \end{cases}$$

Defination 13. Let $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}] \in I(\mathbb{R}_+)$ the interval I is called weakly better then J if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \geq \overline{J}$.

Defination 14. An interval vector $\boldsymbol{z}(\boldsymbol{s}) = (I_i)_{i \in Supp(\boldsymbol{s}) I_i \in I(\mathbb{R})}$, for any $\boldsymbol{s} \in L(N)$is said to be a interval payoff vector for the game $v \in IFG(N)$ if it satisfies the following conditions.

1. $I_i = [0,0], \forall i \notin Supp(\boldsymbol{s})$,
2. $\sum_{i \in Supp(\boldsymbol{s})} I_i = v(\boldsymbol{s})$,
3. I_i is weakly better than $s_i . v(\{i\}), i \in Supp(\boldsymbol{s})$.

Defination 15. Let for $v \in IFG(N)$ and $\boldsymbol{s} \in L(N)$, an interval vector $\boldsymbol{z}(\boldsymbol{s}) = (I_i)_{i \in Supp(\boldsymbol{s}) I_i \in I(\mathbb{R})}$ is said to be a fuzzy population monotonic interval allocation scheme $FPMIAS$ if $\sum_{i \in Supp(\boldsymbol{s})} I_i = v(\boldsymbol{s})$ for each $\boldsymbol{s} \in L(N)$, and for $\boldsymbol{z}(\boldsymbol{t}) = (J_i)_{i \in Supp(\boldsymbol{t}) J_i \in I(\mathbb{R})}, \boldsymbol{z}(\boldsymbol{p}) = (L_i)_{i \in Supp(\boldsymbol{p}) L_i \in I(\mathbb{R})}$ and L_i is weakly better than J_i, with $\boldsymbol{t}, \boldsymbol{p} \in L(N)$ and $Supp(\boldsymbol{t}) \subseteq Supp(\boldsymbol{p})$.

Defination 16. The interval valued Shapley function on$IFG'(N) \subseteq IFG(N)$is a function $\psi: IFG'(N) \to I(\mathbb{R}_+)^N$ that satisfies following four axioms.

Axiom F₁. If $\boldsymbol{s} \in L(N)$and $v \in IFG(N)$ then

$$\sum_{i \in N} \psi_i (\boldsymbol{s}, v) = v(\boldsymbol{s}) \text{ and}$$

$$\psi_i (\boldsymbol{s}, v) = 0 \quad \forall \quad i \notin Supp(\boldsymbol{s}).$$

Axiom F₂. If $v \in IFG'(N)$and $\boldsymbol{s}' \in \boldsymbol{s}^f$ is a fuzzy carrier for $\boldsymbol{s}$in v, then

$$\psi_i (s', v) = \psi_i (s, v) \; \forall \; i \in N,$$

Axiom F₃. If $v \in IFG'(N)$,$\boldsymbol{s} \in L(N)$ and $v(\boldsymbol{t}) = v(\boldsymbol{Y}^{ij}[\boldsymbol{t}])$ for every given $\boldsymbol{t} \in \boldsymbol{s}^f$ and $i, j \in Supp(\boldsymbol{s})$, then

$$\psi_i (\boldsymbol{s}, v) = \psi_j (\boldsymbol{s}, v).$$

Axiom F₄. If $u, v \in IFG'(N)$ and $u(\boldsymbol{t}) = [\underline{u(\boldsymbol{t})}, \overline{u(\boldsymbol{t})}]$, $v(\boldsymbol{t}) = [\underline{v(\boldsymbol{t})}, \overline{v(\boldsymbol{t})}] \in I(\mathbb{R})$, define a game $u + v$such that $(u + v)(\boldsymbol{t}) = [\underline{(u + v)(\boldsymbol{t})}, \overline{(u + v)(\boldsymbol{t})}]$= $[\underline{u(\boldsymbol{t})}, \overline{u(\boldsymbol{t})}]$+$[\underline{v(\boldsymbol{t})}, \overline{v(\boldsymbol{t})}] = u(\boldsymbol{t}) + v(\boldsymbol{t})$ i.e., $(u + v)(\boldsymbol{t}) = u(\boldsymbol{t}) + v(\boldsymbol{t})$ for any $\boldsymbol{t} \in L(\boldsymbol{s})$. If $u + v \in IFG'(N)$and $\boldsymbol{s} \in L(N)$, then

$\psi(\boldsymbol{s}, u + v) = \psi(\boldsymbol{s}, u) + \psi(\boldsymbol{s}, v) \; \forall \; i \in N.$

Defination 17. [9] The game $v \in IFG(N)$ is said to be a fuzzy cooperative interval game with Choquet integral from if and only if for any $\boldsymbol{s} \in L(N)$,

$Q(\boldsymbol{s}) = \{i \in N\colon\ s_i > 0,\ i \in N\}$ and $q(\boldsymbol{s}) = |Q(\boldsymbol{s})|$. We write the elements of $Q(\boldsymbol{s})$in the increasing order as $h_1 < \cdots < h_{q(\boldsymbol{s})}$.

$v(\boldsymbol{s}) = \sum_{l=1}^{q(s)} v([\boldsymbol{s}]_{h_l}).(h_l - h_{l-1})$for any $\boldsymbol{s} \in L(N)$, where $h_0 = 0$, and $[\boldsymbol{s}]_{h_l} = \{i \in N\colon\ s_i \geq h_l\} \in 2^N$ for any $h_l \in Q(\boldsymbol{s})$**.** The set of all fuzzy cooperative interval games with Choquet integral form is denoted by $IFG_C(N)$.

4. Interval valued Shapley function for fuzzy cooperative interval games with Choquet integral form

Wenow deal with the notion of interval valued Shapley functionfor fuzzy cooperative interval games to the class $IFG_C(N)$ of fuzzy games with player set N.

Theorem 4. If $v \in IFG_C(N)$ and $\boldsymbol{s} \in L(N)$, then the function $\psi\colon IFG_C(N) \rightarrow I(\mathbb{R}_+)^N$, defined by

$$\psi_i\,(\boldsymbol{s},\ \boldsymbol{v}) = \sum_{l=1}^{q(s)} \varphi_i([\boldsymbol{s}]_{h_l}, v).(h_l - h_{l-1}) \qquad (4.1)$$

is an interval valued Shapley function in $\boldsymbol{s}$ for $v \in IFG_C(N)$, where

$$\varphi_i([\boldsymbol{s}]_{h_l}, v) = \begin{cases} \sum_{i\in T\subseteq[\boldsymbol{s}]_{h_l}} \dfrac{(|T|-1)!\,(|[\boldsymbol{s}]_{h_l}| - |T|)!}{|[\boldsymbol{s}]_{h_l}|!}\{v(T) - v(T\backslash i)\}\,, & if\ i \in [\boldsymbol{s}]_{h_l} \\ 0, & elsewhere \end{cases}$$

is the interval valued Shapley function of the crisp cooperative interval game $v \in SMIG(N)$.

Proof. We have to verify that the function ψ defined by the formula (4.1) satisfies Axiom **F₁-F₄** of interval valued Shapley function for fuzzy cooperative interval games.

AxiomF₁.

For any$v \in IFG_C(N)$and $\boldsymbol{s} \in L(N) \Rightarrow [\boldsymbol{s}]_{h_l} \in 2^N$, so axiom **F₁** canbe used for interval valued Shapley function under $SMIG(N)$. Hence $\sum_{i\in[\boldsymbol{s}]_{h_l}} \varphi_i([\boldsymbol{s}]_{h_l}, v) = v([\boldsymbol{s}]_{h_l})$ holds for any$l \in \{1, \dots, q(\boldsymbol{s})\}$, we obtain

$$\sum_{i\in N} \psi_i\,(\boldsymbol{s},\ v) = \sum_{l=1}^{q(s)} \sum_{i\in N} \varphi_i([\boldsymbol{s}]_{h_l}, v).(h_l - h_{l-1})$$

$= \sum_{l=1}^{q(s)} v\big([\boldsymbol{s}]_{h_l}\big).(h_l - h_{l-1}) = v(\boldsymbol{s}).$

Since $i \notin Supp(\boldsymbol{s}) \Rightarrow i \notin [\boldsymbol{s}]_{h_l}$, we must have $\varphi_i\big([\boldsymbol{s}]_{h_l}, v\big) = 0.$

It follows that, $\psi_i(\boldsymbol{s}, v) = \sum_{l=1}^{q(s)} \varphi_i\big([\boldsymbol{s}]_{h_l}, v\big).(h_l - h_{l-1}) = 0.$

AxiomF₂.

Given an interval fuzzy game $v \in IFG_C(N)$and $\boldsymbol{s} \in L(N)$ and $\boldsymbol{s}' \in \boldsymbol{s}^f$is a fuzzy carrier for $\boldsymbol{s}$ in$v$.

Let`$P([\boldsymbol{s}]_h) = \{[\boldsymbol{t}]_h : \boldsymbol{t} \in \boldsymbol{s}^f\} \forall\ h \in (0,1]$. Along the line of Tsurumi [9] we have the following. $\boldsymbol{s}' \in S_C(v, \boldsymbol{s}) \Leftrightarrow v(\boldsymbol{s}' \wedge \boldsymbol{t}) = v(\boldsymbol{t})\ \forall\ \boldsymbol{t} \in \boldsymbol{s}^f \Leftrightarrow$ $v([\boldsymbol{s}' \wedge \boldsymbol{t}]_{\boldsymbol{h}}) = \boldsymbol{v}([\boldsymbol{t}]_h) \forall\ \boldsymbol{t} \in \boldsymbol{s}^f \forall\ h \in (0,1] \Leftrightarrow \boldsymbol{v}([\boldsymbol{s}']_h \cap [\boldsymbol{t}]_h) =$ $\boldsymbol{v}([\boldsymbol{t}]_h) \forall\ \boldsymbol{t} \in \boldsymbol{s}^f\ \forall\ h \in (0,1] \Leftrightarrow v([\boldsymbol{s}']_h \cap K = v(K) \forall\ K \in P([\boldsymbol{s}]_h)\ \forall\ h \in$ $(0,1] \Leftrightarrow [\boldsymbol{s}']_h \in C([\boldsymbol{s}]_h | v) for\ any\ h \in (0,1]$. By Axiom **S₂**, $\varphi_i([\boldsymbol{s}]_h, v) =$ $\varphi_i([\boldsymbol{s}']_h, v) for\ any\ h \in (0,1]$. Hence we obtain $\psi_i(\boldsymbol{s}, v) = \psi_i(\boldsymbol{s}', v)$.

AxiomF₃.

If $v \in IFG_C(N)$and $\boldsymbol{s} \in L(N)$. We have the following. $v(\boldsymbol{t}) - v(\boldsymbol{\Upsilon}^{ij}[\boldsymbol{t}]) =$ $0\ \ \forall\ \boldsymbol{t} \in \boldsymbol{s}^f$, such that $t_j = 0$ and $t_k \in \{t_i, 0\} \forall\ k \in Supp(\boldsymbol{s}) \Leftrightarrow v(\boldsymbol{t}) -$ $v(\boldsymbol{\Upsilon}^{ij}[\boldsymbol{t}]) = 0\ \ \forall\ \boldsymbol{t} \in \boldsymbol{s}^f$, such that $t_i = h$ and $t_j = 0$ and $t_k \in \{h, 0\} \forall\ k \in$ $Supp(\boldsymbol{s}), \forall\ h \in (0, s_i] \Leftrightarrow \{v([\boldsymbol{t}']_{\boldsymbol{h}} \cup \{i\}) - v([\boldsymbol{t}']_{\boldsymbol{h}} \cup \{j\})\}.h =$ $0, \forall\ \boldsymbol{t}' \in \boldsymbol{s}'^f,\ such\ that t'_i = t'_j = 0$ and $t'_k \in \{h, 0\}\ \forall\ k \in Supp(\boldsymbol{s}), \forall\ h \in$ $(0, s_i] \Leftrightarrow \{v(T \cup \{i\}) - v(T \cup \{j\})\} = 0\ \ \forall\ T \in P([\boldsymbol{s}]_h \backslash \{i, j\})\ \ \forall\ h \in$ $(0, s'_i]$.

Consiquently, if $v(\boldsymbol{t}) = v(\boldsymbol{\Upsilon}^{ij}[\boldsymbol{t}])\ \ \forall\ \boldsymbol{t} \in \boldsymbol{s}^f$then $v(T \cup \{i\}) =$ $v(T\{j\})\ for\ any\ T \in P([\boldsymbol{s}]_h \backslash \{i, j\})\ \ and\ \ h \in (0,\ s_i]$. Hence we have $\varphi_i([\boldsymbol{s}]_h, v) = \varphi_j([\boldsymbol{s}]_h, v),\ \ for\ any\ h \in (0,\ s_i]\ and\ \varphi_i([\boldsymbol{s}]_h, v) =$ $\varphi_j([\boldsymbol{s}]_h, v) = 0, for\ any\ h \in (s_i, 1]$.

Therefore,$\varphi_i([\boldsymbol{s}]_h, v) = \varphi_j([\boldsymbol{s}]_h, v), for\ any\ h \in (0,1]$. It follows that $\psi_i(\boldsymbol{s}, v) = \psi_j(\boldsymbol{s}, v)$.

AxiomF₄.

If $u, v \in IFG_C(N)$and $\boldsymbol{s} \in L(N)$.

$(u + v)(\boldsymbol{s}) = \sum_{l=1}^{q(s)} (u + v)\big([\boldsymbol{s}]_{h_l}\big).(h_l - h_{l-1})$

$= \sum_{l=1}^{q(s)} u\big([\boldsymbol{s}]_{h_l}\big).(h_l - h_{l-1}) + \sum_{l=1}^{q(s)} v\big([\boldsymbol{s}]_{h_l}\big).(h_l - h_{l-1})$

$= u(s) + v(s) \Rightarrow (u + v) \in IFG_C(N)$.

Since φ is additive. So for any $u, v \in IFG_C(N)$, $\psi_i(\boldsymbol{s}, u+v) = \sum_{l=1}^{q(s)} \varphi_i([\boldsymbol{s}]_{h_l}, u+v).(h_l - h_{l-1})$

$\Rightarrow \psi_i(\boldsymbol{s}, u+v) = \varphi_1([\boldsymbol{s}]_{h_1}, u+v).(h_1 - h_0) + \varphi_2([\boldsymbol{s}]_{h_2}, u+v).(h_2 - h_1) + \cdots + \varphi_{q(s)}\left([\boldsymbol{s}]_{h_{q(s)}}, u+v\right).\left(h_{q(s)} - h_{q(s)-1}\right).$

$\Rightarrow \psi_i(\boldsymbol{s}, u+v) = \{\varphi_1([\boldsymbol{s}]_{h_1}, u).(h_1 - h_0) + \varphi_2([\boldsymbol{s}]_{h_2}, u).(h_2 - h_1) + \cdots + \varphi_{q(s)}\left([\boldsymbol{s}]_{h_{q(s)}}, u\right).\left(h_{q(s)} - h_{q(s)-1}\right)\} + \{\varphi_1([\boldsymbol{s}]_{h_1}, v).(h_1 - h_0) + \varphi_2([\boldsymbol{s}]_{h_2}, v).(h_2 - h_1) + \cdots + \varphi_{q(s)}\left([\boldsymbol{s}]_{h_{q(s)}}, v\right).\left(h_{q(s)} - h_{q(s)-1}\right)\}$

$\Rightarrow \psi_i(\boldsymbol{s}, u+v)$

$$= \sum_{l=1}^{q(s)} \varphi_i([\boldsymbol{s}]_{h_l}, u).(h_l - h_{l-1}) + \sum_{l=1}^{q(s)} \varphi_i([\boldsymbol{s}]_{h_l}, v).(h_l - h_{l-1})$$

$$\Rightarrow \psi_i(\boldsymbol{s}, u+v) = \psi_i(\boldsymbol{s}, u) + \psi_i(\boldsymbol{s}, v).$$

Example 1. Let $N = \{1,2,3\}$ and $\boldsymbol{s} = (0.1, 0.3, 0.5) \in L(N)$. The interval crisp game $v \in SMIG(N)$ is defined as follows. $v(\{1\}) = [0,0]$, $v(\{2\}) = v(\{3\}) = [2,4]$, $v(\{1,2\}) = v(\{1,3\}) = [3,6]$, $v(\{2,3\}) = [5,8]$, $v(\boldsymbol{s}) = [10,20]$. The interval valued Shapley function $\varphi(\{1,2,3\}, v) = \left(\left[\frac{4}{3}, \frac{8}{3}\right], \left[\frac{7}{3}, 4\right], \left[\frac{19}{3}, \frac{40}{3}\right]\right)$, $\varphi(\{2,3\}, v) = ([1,2],[2,4])$, $\varphi(\{1,3\}, v) = ([2,4],[3,4])$ and $\varphi(\{3\}, v) = [2,4]$. With $\boldsymbol{s} \in L(N)$ given as above $v \in IFG_C(N)$, we obtain the worth of $\boldsymbol{s}$under a interval fuzzy game in Choquet integral form namely $v(\boldsymbol{s}) = \sum_{l=1}^{q(s)} v([\boldsymbol{s}]_{h_l}).(h_l - h_{l-1})\, v([\boldsymbol{s}]_{0.1})(0.1) + v([\boldsymbol{s}]_{0.3})(0.2) + v([\boldsymbol{s}]_{0.5})(0.2) = [2.4, 4.4]$. Now the interval valued Shapley function for interval fuzzy game in Choquet integral form is $\psi_i(\boldsymbol{s}, v) = \sum_{l=1}^{q(s)} \varphi_i([\boldsymbol{s}]_{h_l}, v).(h_l - h_{l-1}) = \varphi_i([\boldsymbol{s}]_{0.1}, v)(0.1) + \varphi_i([\boldsymbol{s}]_{0.3}, v)(0.2) + \varphi_i([\boldsymbol{s}]_{0.5}, v)(0.2)$. Therefore $\psi_1(\boldsymbol{s}, v) = \varphi_1([\boldsymbol{s}]_{0.1}, v)(0.1) + \varphi_1([\boldsymbol{s}]_{0.3}, v)(0.2) + \varphi_1([\boldsymbol{s}]_{0.5}, v)(0.2) = \left[\frac{4}{3}, \frac{8}{3}\right](0.1) + [0,0](0.2) + [0,0](0.2) = \left[\frac{0.4}{3}, \frac{0.8}{3}\right]$, $\psi_2(\boldsymbol{s}, v) = \left[\frac{7}{3}, 4\right](0.1) + [2,4](0.2) + [0,0](0.2) = \left[\frac{1.9}{3}, 1.2\right]$ and $\psi_3(\boldsymbol{s}, v) = \left[\frac{19}{3}, \frac{4}{3}\right](0.1) + [3,4](0.2) + [2,4](0.2) = \left[\frac{4.9}{3}, \frac{8.8}{3}\right]$. Hence $\psi\ (\boldsymbol{s}, v) = \left(\left[\frac{0.4}{3}, \frac{0.8}{3}\right], \left[\frac{1.9}{3}, 1.2\right], \left[\frac{4.9}{3}, \frac{8.8}{3}\right]\right)$.

5. Conclusion

In this paper we have characterized the interval valued Shapley function for interval fuzzy game. The properties are direct consequences of their counterparts in crisp games. We have researched interval valued Shapley function for interval fuzzy games in Choquet integral form. Uniqueness of interval valued Shapley function using the class $IFG_C(N)$ will be studied as part of our future research work.

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