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FAULT-TOLERANT METRIC BASES FOR SQUARE OF PATHS

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Abstract

A pair of vertices u and v in a graph G is said to be resolved by the vertex w if the distance between u and w is not equal to the distance between v and w symbolically we write $d(u,w) \neq d(v,w)$. For a simple connected graph G, a set of vertices R of G is said to be a resolving set of G if every pair of vertices of G are resolved by some vertices in R, i.e., every pair of vertices of G are uniquely identified by some vertex elements in R. The resolving set of G containing the minimum number of vertices is the metric basis and the minimum cardinality of the metric basis is called the metric dimension of G. A resolving set F for the graph G is said to be fault tolerant if for each $u \in F$, $F \setminus \{u\}$ is also a resolving set for G and the minimum cardinality of the fault-tolerant resolving set is the fault-tolerant metric dimension. In this article, we study the fault-tolerant metric dimension for P_n^2 for all $n \geq 5$. We have successfully deduced the fault-tolerant metric dimension of P_n^2 for all $n \geq 5$. The existence of at least n fault-tolerant metric bases for the same graph has been ascertained in this paper.

Keywords: Fault-tolerant resolving set, resolving set, metric basis.

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1. Introduction

Every network can be represented by some graph. The identification of every vertex (node) uniquely is of great prominence so as to maintain the security of the network. Now the questionthat needs to be posed is "What should be an identifying method for a given graph?". The distances in graphs play a vital role for the identification of vertices uniquely when the graphis connected. Let $G = \big(V\,(G), E(G)\big)$ be a simple connected graph and $R = \{r_1, r_2, \ldots, r_m\}$ be a set of vertices with respect to R, we define the codes for each vertex of G as follows

$$code_R(v) = (d(r_1, v), d(r_2, v), \ldots, d(r_m, v)),$$

where $d(u, v)$ denotes the distance between the vertices u and v in G. It is to be noted that the code of the vertex v with respect to $R \subset V\,(G)$ is a vector with $|R|$components. A natural intuition can be established that R can identify all vertices uniquely if $codeR(v) \neq codeR(w)$ for every pair of vertices v and w. Such type of set R is called a resolving set. Consequently, a set of vertices R of G is said to be a resolving set if $codeR(v) \neq codeR(w)$ every pair of vertices v and w. The metric basis for a graph G is the resolving set of G containing the minimum number of vertices. The minimum cardinality $\beta(G)$ of the resolving set R is called the *metric dimension* of G and denoted is by $\beta(G)$. A fault-tolerant resolving set with minimum cardinality is called *fault-tolerant metric basis* for G. The concept of metric dimensions was first instigated by Slater [1], Harry and Melter [2].The metric basis $\beta(G)$ is the minimum cardinality of theresolving set. Elements in the basis were considered as sensors in an application given in [2]. The problem of finding the metric dimension for a general graph is a NP-hard. Khuller et al.[3] gave a construction that proves that the metric dimension of a graph is NP-hard.

Although the applications of metric bases arise in many various platforms such as Robot Navigation, Network Optimization, Sensor networks, heavily used by government organization ofIndia such as DRDO, ISRO etc., but still they have some reservations due to the fact that ifsome detectors (elements of metric basis) are faulty, then it is not possible to identify the nodesuniquely. In order to improve the accuracy of the detection or the robustness of the systemHernando et al. introduced concept of fault-tolerant metric dimension in [7]. This concept is defined as follows: A resolving set F of a graph G is fault-tolerant if $F \setminus \{v\}$ is also a resolving set, for every vertex $v \in F$. The *fault-tolerant metric dimension* of G, denoted by $\beta'(G)$, is the minimum cardinality of a fault-tolerant resolving set. A fault-tolerant resolving set of order $\beta'(G)$ is called a fault-tolerant metric basis. The problem of determining the fault-tolerantmetric dimension is a NP hard problem and results are known only for some classes of graphs. Hernendo et al. [7] characterized all fault tolerant resolving sets for any tree T. In this article they also have shown the relation $\beta'(G) \leq \beta(G)(1 + 2 \cdot 5^{\beta(G)-1})$ for every graph G. For a cycle C_n, the fault-tolerant metric dimension has been determined by Javaid et al. in [11] as $\beta'(C_n) = 3$. Basak at el. [12] determine the fault-tolerant metric dimension of C_n^3. In this paper, we study the fault-tolerant metric dimension of P_n^2. We firstly found out the lower bound for $\beta'(P_n^2)$ and then determine fault tolerant metric bases. Lastly, we were able to determine the exact valuefor the same.

The rest of the paper are organized as follows: In the Preliminaries section, we explicitly define and explain the various different terminologies and expressions with the help of which weestablish the various results for the fault-tolerant metric dimension of P_n^2. In section named as Fault-tolerant metric dimension of P_n^2, we have put forward and proved the different lemmas,theorems and examples to claim our results. In this section we were been able to find that the lower bound for the fault-tolerant metric dimension of P_n^2 for $n \geq 5$ i.e., $\beta'(P_n^2) \geq 4$. We have also successfully established that there exists a fault-tolerant resolving set of P_n^2 with cardinality 4 for $n \geq 5$, which later led to the findings of the upper bound for the fault-tolerant metric dimension of P_n^2 which was found as 4 i.e., $\beta'(P_n^2) \leq 4$. The later part of the article consists of the concluding remark, acknowledgement and the references that we have used to construct the article.

2. Preliminaries

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Two vertices u and v are called *adjacent* if there is an edge between u and v. An edge e is called *adjacent to a vertex* v if e has one end as v. The degree of a vertex $v \in V(G)$ is the number of edges adjacent to v. The distance between two vertices u and v, denoted by $d_G(u, v)$ (or simply $d(u, v)$), is the length of shortest paths between them. A path graph P_n (simply we call path) is an n-vertex graph in which every vertex has degree 2 except two end vertices. Henceforth we denote the vertex set $V(P_n)$ by $\{v_0, v_1, \dots, v_{n-1}\}$ and hence degree of v_ℓ is 2 for all $\ell \in \{1, 2, \dots, n-2\}$, whereas both the vertices v_0 and v_{n-1} has degree one. The square of a connected graph G, denoted by G^2, is the graph on the same vertex set as G and two vertices u and v are adjacent in G^2 if $d_G(u, v) \leq 2$. Therefore, vertex set $V(P_n^2)$ is $\{v_0, v_1, \dots, v_{n-1}\}$ and the following proposition is true for P_n^2.

Proposition 2.1 *The distance between two vertices* v_i *and* v_j *in* P_n^2 *is given by* $d(v_i, v_j) = \left\lceil \frac{|i-j|}{2} \right\rceil$ *and the diameter of* P_n^2 *is* $\left\lceil \frac{n-1}{2} \right\rceil$.

Now we define two sets which forms a partition of the vertex set of P_n^2.

Definition 2.1 A vertex $v_i$in P_n^2 is called an *even or odd vertex* if accordingly, the subscript i being an even or odd integer. Let $S_{[0]}$ and $S_{[1]}$, respectively, denote set of all even and odd vertices of P_n^2. Note that $V(P_n^2) = S_{[0]} \cup S_{[1]}$ is a partition of $V(P_n^2)$. For $t \in \{0, 1\}$, an element $v_j \in V(P_n^2)$ is called the *largest element* of $S_{[t]}$ if j is the largest integer such that $v_j \in S_{[t]}$. By intuition similarly we will define the term second largest element in the set $S_{[t]}$ for $t \in \{0,1\}$. The following lemma gives a basic property of a faut-tolerant resolving set for an arbitrary graph.

Lemma 2.1[11] *A set $F \subset V(G)$ is a fault-tolerant resolving set of G if and only if every pair of vertices in G is resolved by at least two vertices of F .*

3. Fault-tolerant metric dimension of P_n^2

In this section, we determine the fault-tolerant metric dimension of P_n^2. First we give a lower bound for $\beta'(P_n^2)$ and then determine fault tolerant metric bases. We need following results to determine a lower bound for $\beta'(P_n^2)$.

Lemma 3.1 *Let v_j resolves two consecutive vertices v_a and v_{a+1}. Then*
(a) $j \equiv a(mod\ 2)$ *with* $j \leq a$
(b) $j \equiv a+1\ (mod\ 2)$ *with* $j \geq a+1$.

Proof:

(a) Let $v_j \in V(P_n^2)$ with $j \leq a$. Then the distances of two vertices $v_a$and v_{a+1} from $v_j$are $\left\lceil\frac{a-j}{2}\right\rceil$and $\left\lceil\frac{a-j+1}{2}\right\rceil$, respectively. Let $a-j=2q+r$ where $0 \leq r \leq 1$. Then $\left\lceil\frac{a-j}{2}\right\rceil \neq \left\lceil\frac{a-j+1}{2}\right\rceil$ implies $\left\lceil\frac{2q+r}{2}\right\rceil \neq \left\lceil\frac{2q+r+1}{2}\right\rceil$ and this is true only when $r=0$. Therefore $j \equiv a(mod\ 2)$ with $j \leq a$.

(b) Let $v_j \in V(P_n^2)$ with $j \geq a+1$, i.e., v_j be a right-side vertex of v_{a+1}. Then the distances of two vertices v_a and v_{a+1} from v_j are $\left\lceil\frac{j-a}{2}\right\rceil$ and $\left\lceil\frac{j-a-1}{2}\right\rceil$, respectively. Let $j-a-1=2q+r$ where $0 \leq r \leq 1$. Then $\left\lceil\frac{j-a}{2}\right\rceil \neq \left\lceil\frac{j-a-1}{2}\right\rceil$ implies $\left\lceil\frac{2q+r+1}{2}\right\rceil \neq \left\lceil\frac{2q+r}{2}\right\rceil$ and this is true only when $r=0$. Therefore $j \equiv a+1(mod\ 2)$ with $j \geq a+1$.

Lemma 3.2 *If F misses an element from $\{v_1, v_2, \ldots, v_{n-2}\}$, then* $|F| \geq 4$.

Proof: Let $v_i \notin F$for some $i \in \{1,2,\ldots,n-2\}$. Then consider three consecutive vertices u_{i-1}, u_i, u_{i+1}. For each $\ell \in \{i, i+1\}$, let R_ℓ denotes the set of vertices which resolves $u_{\ell-1}$ and u_ℓ. Then from Lemma 3.1, $R_i = \{j \equiv i-1(mod\ 2): 0 \leq j \leq i-1\} \cup \{j \equiv i(mod\ 2) : i \leq j \leq n-1\}$and $R_{i+1} = \{j \equiv i\ (mod\ 2) : 0 \leq j \leq i\} \cup \{j \equiv i+1\ (mod\ 2) : i+1 \leq j \leq n-1\}$. Since F bea fault-tolerant resolving set, $|F \cap R_i| \geq 2$ and $|F \cap R_{i+1}| \geq 2$. It is clear that $R_i \cap R_{i+1} = \{u_i\}$.Since $v_i \notin F, |F \cap (R_i \backslash \{v_i\})| \geq 2$ and $|F \cap (R_{i+1} \backslash \{v_i\})| \geq 2$. Since $(R_i \cap R_{i+1}) \setminus \{v_i\} = \emptyset, |F| \geq |F \cap (R_i \backslash \{v_i\})| + |F \cap (R_{i+1} \backslash \{v_i\})| \geq 4$.

Theorem 1*For a square of path P_n^2 with $n \geq 5$, $\beta'(P_n^2) \geq 4$.*

Proof: Let F be an arbitrary fault-tolerant resolving set of P_n^2. If F miss a vertex v_i for some $i \in \{1, 2, \dots, n-2\}$, then applying Lemma 3.2, we have $|F| \geq 4$. Again if F does not miss any vertex from $\{v_1, v_2, \dots, v_{n-2}\}$, then $|F| \geq n-2$. Therefore, $|F| \geq min\{4, n-2\}$ and hence the result is true as F being an arbitrary fault-tolerant resolving set of P_n^2.

Theorem 2 *For every integer $n \geq 5$, there exists a fault-tolerant resolving set of P_n^2 with cardinality* 4.

Proof: For the existence of a fault-tolerant resolving set having cardinality 4, we need to construct a set F for which every pair of vertices of P_n^2 must be resolved by at least two elements of F. The first four consecutive vertices or the last consecutive vertices do have this property. Let us consider $F = \{v_0, v_1, v_2, v_3\}$, i.e., F be the set of first four consecutive vertices. We show that F is a fault-tolerant resolving set. Let u and v be arbitrary two vertices in $V(P_n^2)$. Without loss of generality, we may assume u is a left side vertex of v. Then there exist i and j with $i < j$ such that $u = v_i$ and $v = v_j$. If both of u and v are in F, then the pair of vertices u and v are resolved by both of them. So, in this case proof is trivial. We take the following two remaining cases.

Case 1: Exactly one of uand v is in F . Since $u = v_i$ and $v = v_j$ with $i < j$, so in this case v cannot be in F; otherwise, u will also be in F as F contains consecutive vertices starting from initial vertex v_0. Since $u \in F$, the pair of vertices u and v is resolved by a vertex u. Now we have to find another vertex $w \in F$ that will resolve the vertices u and v. In the below we give the distances of u and v from the both vertices v_0 and v_1.

$$d(v_0, u) = d(v_0, v_i) = \left\lceil \frac{i}{2} \right\rceil$$
$$d(v_1, u) = d(v_1, v_i) = \left\lceil \frac{i-1}{2} \right\rceil$$
$$d(v_0, v) = d(v_0, v_j) = \left\lceil \frac{j}{2} \right\rceil$$
$$d(v_1, v) = d(v_1, v_j) = \left\lceil \frac{j-1}{2} \right\rceil.$$

.

Since $u = v_i \in F, 0 \leq i \leq 3$ and then the above equations give $d(v_\ell, u) \leq 2$ and $d(v_\ell, v) \geq 3$ provided $j \geq 6$. Thus, the pair $u = v_i$ and $v = v_j$ are resolved by two vertices v_0 and v_1 which are elements of F, provided $j \geq 6$. Now we take $j \leq 6$. Since $v = v_j \notin F\{v_0, v_1, v_2, v_3\}$, $j \in \{4, 5\}$. Now for $j \in \{4, 5\}$, i.e., $v_j \in \{v_4, v_5\}$, we calculate a 4×6 matrix D_F in which $(i+1)$-th column represents the code of the vertex $v_i$with respect to F, where $0 \leq i \leq 5$.

$$D_F = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

From the above matrix it is clear that every pair of columns has at least two distinct correspondingentries and hence $u = v_i$ and $v = v_j$ are resolved by at least two elements of F, when $v_i \in F$and $v_j \in \{v_4, v_5\}$.

Case 2: Both u and v are not in F. Recall that $u = v_i$ and $v = v_j$, where $i < j$. Let $i \equiv a(mod\ 2)$, where $a \in \{0, 1\}$. Then both v_a and v_{a+2} are in F. Since $v_\ell \in F = \{v_0, v_1, v_2, v_3\}$ and both v_i and v_j are outside of F, $\ell < i < j$. We show that $d(v_\ell, v_i) \neq d(v_\ell, v_j)$ for each $\ell \in \{a, a+2\}$. Note that $\ell \equiv a\ (mod\ 2)$. Here below, we calculate the distances of v_i and v_j from v_ℓ for each $\ell \in \{a, a\ +\ 2\}$.

$$d(v_\ell, v_i) = \left\lceil \frac{i\ -\ \ell}{2} \right\rceil = \frac{i\ -\ \ell}{2}$$

$$\begin{aligned} d(v_\ell, v_j) &= \left\lceil \frac{j\ -\ \ell}{2} \right\rceil \\ &= \left\lceil \frac{i - \ell + j\ -\ i}{2} \right\rceil \\ &\geq \frac{i\ -\ \ell}{2} + 1\ (as\ i - \ell\ \equiv 0\ (mod\ 2)) \\ &= d(v_\ell, v_i) + 1 \end{aligned}$$

Therefore, we have $d(v_\ell, v_i) \neq d(v_\ell, v_j)$ for each $\ell\ \in \{a, a+2\}$. Hence if i is even, then the pair of vertices $u = v_i$ and $v = v_j$ are resolved by the both v_0 and v_2 whereas for odd i , the pair v_i and v_j are resolved by two elements v_1 and v_3.

On account of Case 1 and Case-2, finally we have that every pair of vertices of P_n^2 are resolved by at least two elements of $F = \{v_0, v_1, v_2, v_3\}$ and this thus grantees the existence of a fault-tolerant resolving set for P_n^2 with cardinality 4.

Remark 3.1 From above theorem we may conclude that $\beta'(P_n^2) \leq 4$ for $n \geq 5$.

Remark 3.2 Renaming the vertices of P_n^2 by $w_i = v_{n-1-i}$, one can show that $F = \{w_0, w_1, w_2, w_3\}$ (the set of last four consecutive vertices of P_n^2) forms a fault-tolerant resolving set for P_n^2 for $n \geq 5$.

Remark 3.3 By similar argument as used in the proof of Theorem 2, one can show that any four consecutive vertices form a fault-tolerant resolving set for P_n^2 for $n \geq 5$.

Theorem 3. *For any integer* $n \geq 5$, $\beta'(P_n^2) = 4$.

Proof: If $n \geq 6$, then from Theorem 1 and Remark 3.1, we have $\beta'(P_n^2) = 4$. Therefore thetheorem is true for $n \geq 6$. Now we take $n = 5$. Then from Theorem 1, $\beta'(P_5^2) \geq 3$ and the equality occurs only if $F = \{v_1, v_2, v_3\}$ from a fault-tolerant resolving set of P_n^2. We show that F does not form a fault-tolerant resolving set for P_n^2. We consider a 3×5 matrix D_F in which $(i+1)$-th column represents the code of $v_i$with respect to F for each $i \in \{0, 1, 2, 3, 4\}$.

$$D_F = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \end{pmatrix}$$

In the above matrix, we see that the entries in 1st and 5th columns differ by only one place and hence $F = \{v_0, v_1, v_2\}$ cannot be a fault-tolerant resolving set of P_5^2. Therefore, $\beta'(P_5^2\ ) =\ 4$.
The proof is complete.

Example 3.1 Here we calculate the codes of each vertex for P_{10}^2 with respect to the fault-tolerant resolving set $F = \{v_0, v_1, v_2, v_3\}$.

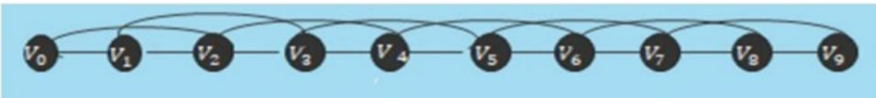

Figure 1: The graph P_{10}^2.

The distance matrix $D$for P_{10}^2 is given by

$$D = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\ 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\ 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 1 \\ 6 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

In the matrix D, the (i,j)-th entry represents the distance between v_i and v_j. Now if we choose a 4×10 sub-matrix D_F consisting of first four rows, we have the following matrix whose $(j+1)$-th column represents the code of v_j with respect to the set $F = \{v_0, v_1, v_2, v_3\}$.

$$D_F = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} c(v_0) & c(v_1) & c(v_2) & c(v_3) & c(v_4) & c(v_5) & c(v_6) & c(v_7) & c(v_8) & c(v_9) \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \end{pmatrix} \end{array}$$

In the above matrix D_F, every pair of columns is different at two places and hence F is a fault-tolerant resolving set. For example, if we take 6-th and 7-th column, then these two columns are differed in 2nd and 4-th places. Again, if we choose a 3×10 sub-matrix consisting of first three rows, we have the following matrix whose (j +1)-th column represents the code of v_j with respect to the set $F' = \{v_0, v_1, v_2\}$.

$$D_{F'} = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} c(v_0) & c(v_1) & c(v_2) & c(v_3) & c(v_4) & c(v_5) & c(v_6) & c(v_7) & c(v_8) & c(v_9) \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \end{pmatrix} \end{array}$$

Here the 6-th and 7-th column are different at only one place. So, F'cannot be fault-tolerant resolving set for P_n^2.

4. Concluding Remark

In this article we have determined the fault-tolerant metric dimension for P_n^2 for all $n \geq 5$. We also have shown the existence of at least n fault-tolerant metric bases for the same graph. The readers may try to find all fault-tolerant metric bases for r-th power of paths or in particular, for square of paths. This article gives a solution to the problem of placement of optimal numbers of sensors in a network when it is structured as square of paths. By giving more than one fault-tolerant metric bases for P_n^2, we present an alternative placement of sensors in the network when one solution is not suitable for an organization who are planning to place the sensors. If a sensor fails which can be catastrophic the fault-tolerant system is able to use reversion to fallback to a safer mode. The advantages of using such a system are that it reduces redundancy, there is no slowdown of the given system and no assumptions are made for the distribution of fault.

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References

[1] Slater, P.J., (1975), Leaves of trees, Cong. Numer., 14, 549–559.

[2] Harary, F., and Melter R.A. (1976), On the metric dimension of a graph, Ars. Combin., 2, 191–195.

[3] Khuller, S., Raghavachari, B., and Rosenfeld, A. (1996), Landmarks in graphs, Discrete Applied Mathematics, 70, 217–229.

[4] Beerliova, Z., Eberhard, F., Erlebach, T., Hall, Hoffmann, A., M., Mihalak, M., and Ram, L., (2006), Network discovery and verification, IEEE J. Sel. Areas Commmun., 24, 2168–2181.

[5] Chartrandand, G., and Zhang, P. (2003), The theory and application of resolvability in graphs, A Survey. Congressus Numerantium., 160, 47–68.

[6] Chartrandand, G., Eroh, L., Jhonson, A., M., and Oellermann, R., O., (2000), Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math., 105, 99–113.

[7] Hernando, C., Mora, M., Slater, J., P., and Wood, R., David.(2008), Fault-Tolerant Metric Dimension of Graphs, Proc. Internat. Conf. Convexity in Discrete Structures, Ramanujan Math. Society Lecture Notes., 5, 81–85.

[8] Imran, M., Baig, A., Bokhary, U., A., S., and Javaid, I., (2012), On the metric dimension of circulant graphs, Applied Mathematics Letters., 25(3), 320–325.

[9] Borchert, A., and Gosselin, S., (2018), The metric dimension of circulant graphs and Caley hypergraphs, Util. Math., 106, 125–147.

[10] C., Grigorious, T., Kalinowski, J., Ryan, and S., Stephen, (2017), The metric dimension of the circulant graph $C(n, 1, 2, 3, 4)$, Australasian journal of combinatories., 69(3), 417–441.

[11] Javaid,I., Salman, M., Chaudhry, A., M. , and Shokat., S., (2009), Fault-Tolerance in Resolvability, Util. Math. 80, 263–275.

[12] Basak, M., Saha, L., Das., K., G., Tiwary., K., (2020), Fault-tolerant metric dimension of circulant graphs $C_n(1, 2, 3)$, Theor.Comput.Sci. 817, 66–79.

[13] Borchert., A., and Gosselin., S., (2018) The metric dimension of circulant graphs and Caleyhypergraphs, Util. Math.,106, 125–147.

[14] C., Grigorious., T., Kalinowski., J., Ryan., and S., Stephen., (2017), The metric dimension of the circulant graph $C(n, 1, 2, 3, 4)$, Australasian journal of combinatories., 69(3), 417–441.