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A NOVEL TECHNIQUE FOR SOLVING MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEMS

Mediya B. Mrakhan[1], Ayad M. Ramadan[2], Rzgar F. Mahmood[3], and Ronak M. Abdullah[4]

[1]Mathematics, College of Education, Garmian University, Kalar, Iraq

[2]Mathematics, College of Science, Sulaimani University, Sulaimani, Iraq

[3]Mathematics, College of Education, Garmian University, Kalar, Iraq

[4]Mathematics, College of Science, Sulaimani University, Sulaimani, Iraq

Email: [1]medya.bawaxan@garmian.edu.krd, [2]ayad.ramadan@univsul.edu.iq
[3]rzgar.fariq@garmian.edu.krd, [4]runak.abdullah@univsul.edu.iq

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Abstract

In this paper we presented a novel method to solve multi-objective linear programming problems (MOLPP) which depends on the scaling method. The new idea is to construct a matrix from some points, then reduce the dimensions of the matrix using cluster method and according to the positions of these points in R^2. The new points have a property that they are in equal distances to the origin point (0, 0). An algorithm is presented to explain the procedure of the method. The experimental results show the effectiveness of this method comparing with other methods. Also, when the optimal solution for each objective functions achieved in different extreme points, a new procedure was introduced to find best solution to help the decision maker in his search.

Keywords: Multi-objective linear programming, scaling method, convex combination.

2010 AMS classification: 90C05

1. Introduction

Sen found out a new approach for multi-objective linear programming problems and suggests an approach to construct the multi-objective function under the limitation that the optimum value of individual problem was greater than zero [8]. After that many researchers attempted to solve multi-objective linear programming problems. Sulaiman and Sadiq studied the multi-objective function by solving the multi-objective programming problem, using mean and median value [11]. Sulaiman and Hamadameen used optimal transformation technique to solve multi-objective linear programming problem [12]. Sulaiman et al. applied geometric average technique to solve extreme point multi-objective quadratic programming problems [14]. Zahidul and Asadujjaman introduced a proposed new average method for solving multi-objective linear programming problem using various kinds of mean techniques [15]. Sen used a statistical approach to solve multi-objective programming (MOPP) problems [9]. All the previous paper doesn't find optimal solution, otherwise give a best solution for the problem, and to choose the best one it is necessary to compare the results.

In this paper we presented a novel technique to find a solution for a multi-objective linear programming problem namely multidimensional scaling which is a technique to represent a similarity or dissimilarity data on a set of objects and attempts to model such data as distances among points in a geometric space, and then find a configuration of the points, preferably in a small number of dimensions, ideally 2 [4]. The results show the effectiveness of this method through some numerical examples. Also, in another case we introduced a new procedure to help the decision maker to find a good solution.

2. Mathematical Modeling of the Multi-Objective Programming Problems (MOPP)

The mathematical model can be formulated to maximize (minimize) several objectives simultaneously subject to some constraints as follows:

$$\left.\begin{array}{ll} Max.\, z_i = c_i{}^t X + \alpha_i & i = 1, \dots, r \\ Min.\, z_i = c_i{}^t X + \alpha_i & i = r+1, \dots, s \end{array}\right\}$$

Subject to:

$$AX \begin{bmatrix} \geq \\ = \\ \leq \end{bmatrix} B \qquad (1.1)$$

$X \geq 0,$

where X is an n-dimensional vector of decision variables c is n-dimensional vector of constants, B is m-dimensional vector of constants, r is the number of objective functions to be maximized, s the number of objective functions to maximized plus minimized, $(s - r)$ is the number of objectives that is to be minimized, A is a $(m \times n)$ matrix of coefficients all vectors are assumed to be column vectors unless transposed, $\alpha_i (i = 1, \dots, s)$ are scalar constants, ${c_i}^t X + \alpha_i \ (i = 1, \dots, s)$ are linear factors for all feasible solutions. The problem is said to be multi-objective linear programming problem (MOLPP) if all the objective functions and constraint functions are linear, and all the variables are continuous variables [13].

3. Multidimensional Scaling (MDS)

Multidimensional scaling starts with proximity between observations to produce their spatial representation [3]. This technique starts with an (k × k) dissimilarity or distance matrix D, with the elements δ_{ij} and d_{ij}, respectively. We wish to represent the k points in a small number of dimensions, in which the distances d_{ij} between items closely match the original δ_{ij}, that is, $d_{ij} = \delta_{ij}$ for all i, j [7]. Many MDS problems can be formulated in terms of the optimization problem. Srinivasan and Shocker proposed a linear programming model for external analysis [10]. Brusco used integer programming methods for Seriation and one-dimensional scaling of proximity matrices [2]. Laeuter and Ramadan used optimization techniques for configuration categorical data [5] and [6].

3.1 The Main Idea

In Sen’s method, firstly all objective functions need to be maximized or minimized individually by Simplex method. The optimal values are

$\text{Max}\, Z_1 = \varphi_1$

$\text{Max}\, Z_2 = \varphi_2$

....

$\text{Max}\, Z_r = \varphi_r$

$\text{Min}\, Z_{r+1} = \varphi_{r+1}$

....

$\text{Max}\, Z_s = \varphi_s$,

These optimal values are used to form a single objective function by adding (for maximum) and subtracting (for minimum) of each result of dividing each z_i by φ_i, where $|\varphi_i| \neq 0$, i.e. Max $Z = \sum_{i=1}^{r} \frac{z_i}{|\varphi_i|} - \sum_{i=r+1}^{s} \frac{z_i}{|\varphi_i|}$ [15] and subject to the same constraints in (1.1). For all the presented methods the goal is to minimize the value of φ_i, different techniques were used to find a value from $\varphi_i (i = 1,2, \dots, s)$, we call this value a divided value and denoted by ρ, it is clear that as much as φ_i is small, the method gives best result.

In a (MOPP) we have s objective functions with some extreme points. To find a fair solution of this problem, there are two techniques; the first one is treats with the optimal values of the s objective functions to find a compromise objective function, say Z, in this case it is not necessary to find the point that gives optimal value of Z, because it is one of the extreme points of the main problem (1.1), as in [4, 8, 11, 12, 14, 15].

The second technique tries to find a new point in the feasible region of (1.1), and then find the optimal value of the s objective functions at this new point. In this case the one find an acceptable best value for the decision maker. The famous paper that used this technique is [16] by Zimmermann, he presented the application of fuzzy linear programming approaches to the linear vectormaximum problem.

In this paper, we presented two new techniques for the two cases. The first technique depends on scaling method. After solving the objective functions individually, we construct a set of ordered pairs by Cartesian product of the absolute values of maximum values and minimum values, and then plot these points in R^2. This technique visualizes the values of the optimal values together in R^2. Choose the points that construct a big cluster, i.e. the number of data within a cluster can be high, say k_1 points. Our start point is matrix D_1 where each row represents a point. The dimensions of D_1 are $(k_1 \times 2)$. To represent these points in a small number of dimensions (2 dimensions) we have to reduce the dimensions of D_1 into (2×2), also use the idea of cluster and choose the big one.

Now we have a matrix D_1 with (2×2) dimensions, use method of multidimensional scaling [7] to find the configuration points that give reasonably good approximations to the distances between the rows of D_1 . The divided factor ρ is the distance between the origin point and the configuration points which is constant for all the points, as illustrates in the following algorithm

Step 1: Find the Cartesian product of the absolute values of individual values of Max z_i and Min z_i, and then plot them in R^2.
Step 2: Choose the big cluster with k_1 points, and then construct a matrix D_1 with dimensions (k_1 x 2) where each row represents a point in the Cartesian product. Reduce the dimensions of D_1 to (2×2) by choosing the big cluster.
Step 3: Use Euclidian distance to find a matrix $ED = \left[d_{ij}\right]$, i, j = 1, ..., 2.
Step 4: Let $A_1 = -0.5 \times [d_{ij}^2]$
Step 5: $B_1 = (I - \frac{1}{2} J) \times A_1 \times (I - \frac{1}{2} J)$, where I_{2x2} is the Identity matrix and J_{2x2} is the unit matrix.
Step 6: Find the eigenvalues λ_i and the eigenvectors v_i of the matrix B_1.
Step 7: The coordinates of the 2 points in the Euclidean space are given by $s_1 = (\sqrt{\lambda_1} \times v_1, \sqrt{\lambda_2} \times v_2)$ and ρ is the distance between these points and the origin point.
Step 8: end.

It is important to mention that the matrix B_1 is symmetric, positive semi definite and of rank p where $p \leq 2$, and hence it has p non-negative eigenvalues and $(2 - p)$ zero eigenvalues. Therefore the number of non-zero eigenvalues gives the number of eigenvalues required for representing the distances in step 7, ideally 2.

3.2 Numerical Examples for the First Technique

In this section we construct some numerical examples and compare them with other methods.

Example 3.2.1: Solve the following (MOLPP)

$$\text{Max } Z_1 = 4x_1 + 2x_2$$
$$\text{Max } Z_2 = 3x_1 + 6x_2$$
$$\text{Max } Z_3 = -8x_1 + 6x_2$$
$$\text{Min } Z_4 = 5x_1 - 7x_2$$

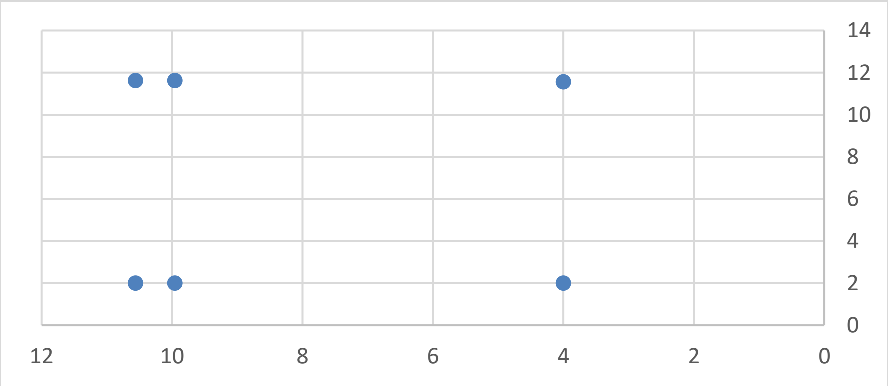
$$\text{Min } Z_5 = 2x_1 - 8x_2,$$

subject to

$$2x_1 + 6x_2 \leq 10$$
$$4x_1 - 2x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Solution:

By simplex method the optimal values are $z_1 = 4$; $z_2 = 10.56$; $z_3 = 9.96$; $z_4 = -11.62$; $z_5 = -2$. So, the Cartesian product is the set {(4, 11.56), (4, 2), (10.56, 11.62), (10.56, 2), (9.96, 11.62), (9.96, 2)}. From the graph bellow we choose the big cluster.



The points in the big cluster are (10.56, 11.62), (10.56, 2), (9.96, 11.62), (9.96, 2).

$$D_1 = \begin{bmatrix} 9.96 & 11.62 \\ 9.96 & 2 \\ 10.56 & 11.62 \\ 10.56 & 2 \end{bmatrix}$$

Reduce the matrix D_1 to the dimensions (2×2). We have two possibilities; we choose one of them arbitrary, say

$$D_1 = \begin{bmatrix} 9.96 & 11.62 \\ 10.56 & 11.62 \end{bmatrix}$$

$$ED = [d_{ij}] = \begin{bmatrix} 0 & 0.6 \\ 0.6 & 0 \end{bmatrix}$$

$$A_1 = -0.5 \times [d_{ij}^2]$$

$$A_1 = \begin{bmatrix} 0 & -0.18 \\ -0.18 & 0 \end{bmatrix}$$

$$B_1 = (I - \frac{1}{2} J) \times A_1 \times (I - \frac{1}{2} J)$$

$$B_1 = \begin{bmatrix} 0.09 & -0.09 \\ -0.09 & 0.09 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0.18$, the corresponding eigenvectors are

$$V_1 = \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix},$$
$$V_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix},$$

respectively. The coordinates of the points are

$$s_1 = \left(\sqrt{0} \times \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix}, \sqrt{0.18} \times \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}\right) = \begin{pmatrix} 0 & -0.3 \\ 0 & 0.3 \end{pmatrix}.$$

So we have two points in R^2 which are (0, - 0.3) and (0, 0.3), and the distance between these points and the origin (0, 0) is 0.3 which is our divided factor ρ.

Example 3.2.2: Solve the following (MOLPP)

$$\text{Max } Z_1 = 2x_1 + x_2 + 5$$
$$\text{Max } Z_2 = 3x_1 + x_2 + 7$$
$$\text{Max } Z_3 = 2x_1 + 2x_2 + 6$$
$$\text{Min } Z_4 = 3x_1 + x_2 + 3$$
$$\text{Min } Z_5 = 3x_1 + 2x_2 + 8$$
$$\text{Min } Z_6 = x_1 + 3x_2 + 5,$$

subject to

$$x_1 + x_2 \geq 1$$
$$3x_1 + 2x_2 \leq 6$$
$$2x_1 + 4x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

In the following table we compare among some methods where ρ the divided value is.

Examples	Correlation method	Optimal geometric average	Optimal average	MDS
Example 3.2.1	ρ = 0.98	ρ =1.42	ρ = 3	ρ = 0.3
Example 3.2.2	ρ = 0.95	ρ =1.57	ρ = 6	ρ = 0.5

4. Convex Combination Approach

Use the definition of the convex combination [1] as a second technique. We introduce the following algorithm

Step 1: Find the feasible region and the extreme points for (1.1), then find $\varphi_i (i = 1,2, \dots, s)$, where $\varphi_i \in$ R.
Step 2: Choose Max. $\{ \varphi_i\}(i = 1,2, \dots, r)$, say it achieves at x^* such that $Z_i(x^*)$ is positive $(i = r + 1, , \dots, s)$, and choose Min. $\{ \varphi_i\}(i = r + 1, , \dots, s)$, say it achieves at y^* such that $Z_i(y^*)$ is positive $(i = 1, \dots, s)$.
If the problem contains only maximum case, i.e., we have only $(i = 1,2, \dots, r)$, then we choose the largest two values of $\{ \varphi_i\}(i = 1,2, \dots, r)$.
Step 3: Find a new point z^* which is a convex combination of x^* and y^*, so $z^* = \sigma_1 x^* + \sigma_2 y^*$, where $\sigma_1, \sigma_2 \geq 0$, and $\sigma_1 + \sigma_2 = 1$. Construct a linear system.
Step 4: Solve the linear system, and the decision maker will choose a best solution according to his case
Step 5: End.

Example 4.1

$$\text{Max}\, Z_1 = -x_1 + 2x_2$$
$$\text{Max}\, Z_2 = 2x_1 + x_2,$$

Subject to

$$-x_1 + 3x_2 \leq 21$$
$$x_1 + 3x_2 \leq 27$$
$$4x_1 + 3x_2 \leq 45$$
$$3x_1 + x_2 \leq 30$$
$$x_1, x_2 \geq 0.$$

Figure 1 shows the feasible region of this problem. The point (0, 7) is optimal with respect to objective function Max $Z_1 = -x_1 + 2x_2 = 14$. Also, the point (9,3) is optimal with respect to objective function Max $Z_2 = 2x_1 + x_2 = 21$.

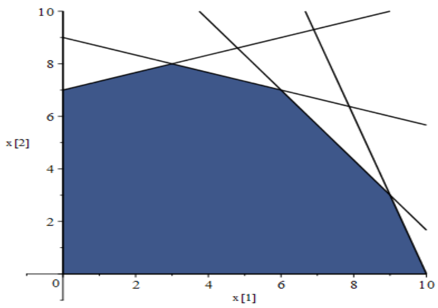


Fig. 1: Graphical solution for the objective function.

To apply our algorithm, at first we find the feasible region, and then the extreme points which are:
(0, 7), (3, 8), (6, 7), (9, 3) and (10, 0), with
Z_1 (0, 7) = 14, Z_1 (3, 8) = 13, Z_1 (6, 7) = 8, Z_1(9, 3) = -3 and Z_1(10, 0) = -10.
Z_2 (0, 7) = 7, Z_2 (3, 8) = 14, Z_2 (6, 7) = 19, Z_2(9, 3) = 21 and Z_2(10, 0) = 20.
We choose the points (0, 7) because it is largest one and Z_2 (0, 7) = 7 is positive. The second point is (6, 7) because it is the largest one and Z_1 (6, 7) = 8 is positive.
Now, to find a new point $z^* = (z_{1,}^* z_2^*)$ use the definition of convex combination, we have the following linear system

$$z_1^* = 6\sigma_2$$
$$z_2^* = 7\sigma_1 + 7\sigma_2$$
$$\sigma_1 + \sigma_2 = 1.$$

The decision maker may test some values to choose the best solution

σ_1	σ_2	z_1^*	z_2^*	Z_1	Z_2
0	1	6	7	8	19
1	0	0	7	14	7

In these two special cases we see that the results are the extreme points, and for other values $0 < \sigma_1, \sigma_2 < 1$, the results lies on the line between (0, 7) and (6, 7). This guarantee that the results are locate in the feasible region. For instances, if $\sigma_1 = 0.1, \sigma_2 = 0.9$, then $Z_1 = 8.6$ and $Z_2 = 17.8$ at $(5.4, 7)$, and if $\sigma_1 = 0.05, \sigma_2 = 0.95$, then $Z_1 = 8.3$ and $Z_2 = 18.4$ at $(5.7, 7)$. By Zimmermann in [16], $Z_1 = 9.61$ and $Z_2 = 17.38$ at $(5.03, 7.32)$.

5. Concluding Remarks

We have studied Multi-objective linear programming problems and found two new techniques to solve this type of problem. The first one found a compromise objective function by using multi-dimensional scaling. The second one used the convex combination idea, and found a new point from the extreme points of the feasible region. In both techniques we presented an algorithm and numerical example to explain the method. Comparing the results with other methods showed the effective of both techniques.

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