Mathematical Forum ISSN: 0972-9852
Vol.28(2), 2020

# CHOICE OF UNCERTAIN PROSPECTS IN THE PRESENCE OF "HUGE" LOSS AVERSION

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Received: 29/05/ 2020

Accepted: 20/12/2020

## Abstract

*We report a thought experiment supported by casual questionnaire conducted on December 14, 2019, which shows that decision making in the presence of a distinct possibility of incurring a "huge" loss may be independent of and not related to the probability distribution over the various states of nature thus leading to a deviation from expected or more generally state-dependent and weighted utility maximization and suggest an alternative solution concept that is similar in spirit to classical decision analysis to provide a realistic solutions for such problems, where the definition of huge loss may be state-dependent.*

**Keywords:** uncertain prospect, expected utility, expected utility maximizer, probabilistic value, probabilistic valuation maximizer, huge loss.

**2010 AMSClassification:** 91B06.

## 1. Introduction

An uncertain (monetary) prospect is a function from a non-empty finite set of mutually disjoint events or states of nature to the set of real numbers, where a real number associated with a state of nature denotes a state-dependent payoff along with a probability distribution over the states of nature. That the states of nature are events (subset of the sample space, even if it were a singleton) and not a sample point is crucial in what follows. The formal definition of an uncertain prospect is available in pages 19 and 20 of Wakker (2010). The utility functions for money we use allow somewhat greater flexibility than those invoked by Wakker (2010), since we allow for state-dependence and the possibility that utility functions may not be strictly increasing for non-positive gains(losses). Being non-decreasing in this region is enough for us.

Consider a real valued random variable X for which the theoretically derived probability distribution can be summarized in a continuous probability density function on an interval in the real line. Such random variables are called continuous random variables. For information about the basic concepts of probability that concern us at this juncture, one may refer to Lahiri (2020). Otherwise Gilboa (2011) is good enough for the rest of the paper. To be specific, the probability distribution of X that we actually used and would recommend for its analytical simplicity is the uniform probability distribution on the closed interval [0,1], although any other continuous probability distribution function with positive support on a non-degenerate interval of the real line (including the real line itself) would serve our purpose equally well. For those unfamiliar with random variables governed by continuous probability density functions or for those who wish to refresh their memory, let us add that the probability of such a random variable assuming a rational number is always zero, though the random variable itself may assume a rational number. Similarly the probability of such a random variable assuming an irrational number is always one, though there is no reason that the prediction must invariably be an irrational number. This kind of phenomena can occur with continuous random variables and is discussed here: https://www.statlect.com/fundamentals-of-probability/zero-probability-events

We now report an observation on December 14, 2019 and reported by the observers in Lahiri and Sikdar (forthcoming). The observation concerns an experiment in which the intended subjects are informed that a real number will be randomly chosen from the closed interval [0,1] and that the probability that the chosen value lies in any sub-interval of [0,1] is equal to the length of the sub-interval, in other words the randomly chosen number X is uniformly distributed on [0,1]. They are informed about the properties of the uniform random variable that are mentioned in the previous paragraph. To emphasize the "equi-probability" aspect the subjects are told that given any two numbers in [0,1] and the same desired level of approximation however close that may be, the approximate probability with which a number is chosen is same for both the numbers. Such a randomization which is convincing for the subjects, could take place in several ways, two of which are mentioned below.

(1) A college mathematics teacher who is a total stranger to the subjects being questioned is asked to "randomly" choose a number in the closed interval [0,1]. The teacher should be given a day or half a day's time to choose the number and communicate it in a sealed envelope to the experimenter, without having to appear before the subjects. The number can be expressed either as a fraction or a decimal number (recurring or otherwise) or as the solution (perhaps implicit) to a problem which has a unique solution in [0,1]. The exponential number 'e' or π is acceptable. So are infinite series such as $\frac{1}{\sqrt{11}}[1 + \frac{1}{1!} + \frac{1}{(2^2)!} + \frac{1}{(3^3)!} + \ldots + \frac{1}{(n^n)!} + \ldots]$. It is not necessary to know the exact number. All that the experimenter is required to do is to recognize whether the number communicated by the college teacher is a rational number or it is an irrational number. This is not the same as allowing the college teacher to announce either "rational number" or "irrational number". This is really all that seems necessary to operationalise the randomization procedure.

Given this information, it seems that the subjects are as justified in concluding that chance that the chosen number lies between 0 and x is twice the chance that it lies between 0 and $\frac{1}{2}$ x as they would be in concluding without further information that each white ball in an urn containing six identical white balls has an equal chance of being chosen as any other if exactly one white ball has to be chosen from the urn.

(2) If one is knowledgeable about modern computational techniques, one may want to look elsewhere for a solution. Chapter 3 of the book in progress by Owen (book in progress) discusses PRNG's (Pseudo-Random Number Generators) for uniform probability distributions.
Before the value of the random variable X is realized, a subject who is well aware of the theoretical results discussed above is offered a choice between the following two uncertain prospects.

Uncertain prospect 1:

If the predicted value of X is an irrational number, the subject wins US$ 500. If the predicted value of X is a rational number, the subject neither wins nor loses any money.

Uncertain prospect 2:

If the predicted value of X is an irrational number, the subject wins US$ 1000. If the predicted value of X is a rational number, the subject **loses** US$ 100 billion.

Which prospect do you think the subject will choose? Which prospect would you choose?

In this case the two states of nature are the events, the predicted value of X is an irrational number and the predicted value of X is a rational number. In the first uncertain prospect, if the first state of nature is realized the pay-off is $500 and if the second state of nature is realized then the pay-off is zero. In the second uncertain prospect, if the first state of nature is realized the pay-off is $1000 and if the second state of nature is realized then the pay-off is $-100 billion.

Most people who were asked this question (including economists, decision theorists, mathematicians, statisticians) chose uncertain prospect 1 in spite of the fact that that the expected monetary value (EMV) of the first is US$500 which is less than the EMV of the second which is US$1000. With a possibly state-dependent utility function for money that is strictly increasing for gains and non-decreasing otherwise, even the expected utility of the second uncertain prospect is greater than the expected utility of the second. Kahneman and Tversky (1979) attribute such behavior to loss aversion. However the behavior we notice here is a very extreme form of (huge?) loss aversion- one associated with a loss whose probability of occurrence is zero. In fact, it would not be remiss to point out, that the observed and plausible responses to the above experiment, conflict with the predictions of any state-dependent and weighted utility maximization theory that requires (a) a possibly state-dependent utility function for money that is strictly increasing for gains and non-decreasing otherwise, and (b) a possibly state-dependent probability weighting function which assigns a weight zero to probability zero and one to probability one. Such a state-dependent weighted utility maximization theory would include Kahneman and Tversky's prospect theory as a special case. Kahneman and Tversky refer to a probability weighting function as a "probability distortion function". A probability distortion function which is assumed to be independent of the state of nature, maps zero to zero, one to one, is initially concave and then convex. That the probability distortion of zero is zero and of one is one implies that according to Kahneman and Tversky's prospect theory, the evaluation of the first prospect is the utility derived from US$500 if the announced number is irrational and the evaluation of the second prospect is the utility derived from US$1000 if the announced number is irrational. With possibly state dependent utility for money that is strictly increasing in gains, the second uncertain prospect has a higher evaluation than the first and so according to prospect theory the second prospect should have been chosen and not the first. This result is inconsistent with the observed responses for the thought experiment discussed above.

Our conjecture is that when faced with huge losses, apart from the magnitude of the loss, what matters are not probabilities but possibilities.Evidence against expected utility maximization under different circumstances abound and has been surveyed lucidly in Sikdar (2006). The well-known Allais Paradox (Allais (1953)) based on thought experiments as well as questionnaire based experimentation is the starting point of this line of investigation. Our combination of thought experiment and questionnaire based experimentation is original because it provides evidence against any reference or consideration of the probability distribution over the states of nature and consequently against state dependent and weighted expected utility maximization, when there is a possibility of "huge" loss.

**2. The model:** We consider a non-empty finite set of possible states of nature $N = \{1,2,\ldots,n\}$ for some positive integer n. A state of nature is an event (i.e. a subset of the sample space) and not a sample point. A possible state of nature is an event which is not the empty set.

**Notation:** Let $\mathbb{R}_+$denote the set of non-negative real numbers $\{\alpha\in\mathbb{R}| \alpha \geq 0\}$ and $\mathbb{R}_-$denote the set of non-positive real numbers $\{\alpha\in\mathbb{R}| \alpha\leq 0\}$.

The decision maker's beliefs about the possible states of nature are summarized in a probability distribution $p = (p_1,\ldots,p_n)\in P^N = \{q = (q_1,\ldots,q_n)\in\mathbb{R}^n_+ |\sum_{i=1}^n q_i = 1\}$.

$P^N$ is said to be the **set of probability distributions** on N.

An **uncertain prospect** is a pair $(x,p)\in\mathbb{R}^n\times P^N$ where $x = (x_1,x_2,\ldots,x_n)$ lists the pay-offs in each and every possible state of nature under consideration.

It is possible to formulate a very general decision theoretic model whose conclusions are contradicted by the plausible choices between uncertain prospects 1 and 2.

A **probabilistic valuation function** is a function $v: \mathbb{R}\times[0,1]\times N\rightarrow \mathbb{R}$, such that for all $i\in N$, (a) $v(\alpha,0,i) = 0$, for all $\alpha\in\mathbb{R}$; (b) the function $v(., 1, i): \mathbb{R}\rightarrow \mathbb{R}$, is assumed to be strictly increasing on $\mathbb{R}_+$ and non-decreasing on $\mathbb{R}_-$.

The **probabilistic value** of an uncertain prospect (x,p) for a decision maker with probabilistic valuation function v, $E[v](x,p) = \sum_{i=1}^n v(x_i, p_i, i)$.

E[v](x,p) is said to be the **probabilistic value** of the uncertain prospect (x,p).

A decision maker with probabilistic valuation function v is said to be a **probabilistic valuation maximizer** if given any non-empty finite set S of uncertain prospects the decision maker chooses an uncertain prospect in S and any such choice $(x,p)\in S$ satisfies $E[v](x,p) \geq E[v](y,q)$ for all $(y,q)\in S$.

It is easy to verify that for any probabilistic valuation function the probabilistic value of the second uncertain prospect is greater than the first uncertain prospect and observed choices are predominantly in favor of the first prospect. Thus such decision makers violate probabilistic valuation maximizing behavior and are thus not probabilistic valuation maximizers.

Classical decision theory is concerned with the following special case of the above decision theory.

The decision maker has a**utility function** for money which is given by a function u:$\mathbb{R}$×N→$\mathbb{R}$ which for all i∈N, is assumed to be strictly increasing on $\mathbb{R}_+$ and non-decreasing on $\mathbb{R}_-$.

In particular the utility function could be the identity function on the real numbers regardless of the state of nature, i.e. u(α,i) = α for all α∈$\mathbb{R}$ and i∈N.

A **probability weight function** is a function π:[0,1]×N→[0,1] such that for all i∈N, π(0,i) = 0 and π(1,i) = 1.

It is possible that for one or more i∈N, it is the case that π(α,i) = α for all α∈N.

The **weighted expected utility** of the uncertain prospect (x,p) for a decision maker with utility function u, and probability weight function π denoted E[u,π](x,p) = $\sum_{i=1}^{n} \pi(p_i, i) u(x_i, i)$.

In particular, if for all i∈N, it is the case that π(α,i) = α for all α∈N, then we have classical expected utility.

A decision maker with utility function u is said to be a **weighted expected utility maximizer** if given any non-empty finite set S of uncertain prospects the decision maker chooses an uncertain prospect in S and any such choice (x,p)∈S satisfies E[u,π](x,p) ≥ E[u,π](y,q) for all (y,q)∈S.

A weighted expected utility maximizer is clearly a probability valuation maximizer, where given any uncertain prospect (x,p), the probability valuation function v satisfies $v(x_i,p_i,i) = \pi(p_i,i)u(x_i,p_i)$ for all i∈N.

**3.Conclusion and an alternative possibility:** The above discussion suggests inconsistency with probabilistic valuation maximization if probabilistic valuation functions are unbounded below.

The implicationof this inconsistency depends to a great extent on the context and how the decision maker decides to define the states of nature. In most decision making problems related to transactions on the market particularly during times when there are no significant social upheavals taking place, states of nature are defined in terms of one or more economic/financial variables. Hence even if there are uncertain prospects with a distinct possibility of yielding a huge loss (as perceived by the decision maker), there usually are uncertain prospects which "keeps clear" of huge losses from which the decision maker can choose from. If this be the case, then the decision maker may simply ignore those prospects which may lead to huge loss(es) and choose from the rest any one that maximizes his/her probabilistic value. This kind of decision making behavior may be formalized as follows.

Given a vector K∈$\mathbb{R}_+^n$ and (x,p)∈$\mathbb{R}^n$×$P^N$, let M(x,K) = {j∈N| $x_j$< -$K_j$}.K is a state-dependent vector of threshold levels for losses, below which an extreme form of "loss aversion" sets in.

A **restricted probabilistic valuation function** is a function v:[-$K_i$, +∞)×[0,1]×N→ $\mathbb{R}$, such that for all i∈N, (a) v(α,0,i) = 0, for all α∈[-$K_i$, +∞); (b) the function v(., 1, i): [-$K_i$, +∞)→ $\mathbb{R}$, is assumed to be strictly increasing on $\mathbb{R}_+$ and non-decreasing on $\mathbb{R}_-$∩[-$K_i$, +∞).

The **probabilistic value** of an uncertain prospect (x,p)$\in(\prod_{i=1}^{n}[-K_i,+\infty))\times P^N$ for a decision maker with restricted probabilistic valuation function v, E[v](x,p) = $\sum_{i=1}^{n} v(x_i, p_i, i)$.

E[v](x,p) is said to be the **probabilistic value** of the uncertain prospect (x,p).

Given a non-empty finite set of uncertain prospects S, let $S_K = \{(x,p)\in S|x_j < -K_j$ for some $j\in N\} = \{(x,p)\in S|M(x,K) \neq \phi\}$.

A decision maker with probabilistic valuation function v is said to be a **probabilistic valuation maximizer** if given any non-empty finite set S of uncertain prospects with $S\backslash S_K\neq\infty$the decision maker chooses an uncertain prospect in $S\backslash S_K$ and any such choice (x,p)$\in S\backslash S_K$ satisfies E[v](x,p) $\geq$ E[v](y,q) for all (y,q)$\in S\backslash S_K$.
In particular, if the decision maker is an expected utility maximize, then it has a restricted utility function (for money) u$[-K_i, +\infty)$:$\times N\rightarrow\mathbb{R}$ which for all $i\in N$, is assumed to be strictly increasing on $\mathbb{R}_+$ and non-decreasing on $\mathbb{R}_-\cap[-K_i, +\infty)$, so that for all uncertain prospects (x,p)$\in(\prod_{i=1}^{n}[-K_i,+\infty))\times P^N$, v satisfies $v(x_i,p_i,i) = \pi(p_i,i)u(x_i,p_i)$ for all $i\in N$.

**Acknowledgment:** We benefitted from astute comments by SurajitBorkotokey, NilotpalChakravarty, SujoyChakrabarti and very enlightening and useful suggestions from Michael Birnbaum, DebkumarChakrabarti, ItzhakGilboa and Manuel Matos. We wish to thank all of them for their valuable comments and suggestions. The thought experiment reported in the first section is available at:http://mail.sjdm.org/pipermail/jdm-society/2020-April/008485.html. However responsibility for all residual errors remain with the authors.

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